

Euclidean Freeze-Tag Problem on Plane*

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Abstract

The freeze-tag problem is an optimization problem introduced by Arkin et al. (SODA'02). This problem revolves around efficiently waking up a swarm of inactive robots starting with a single active robot. Each asleep robot is activated by an awake robot going to its location. The objective is to minimize the total wake-up time for all robots, the *makespan*.

A recent paper by Bonichon et al. considers the geometric version of the freeze-tag problem on the plane. They conjectured that for the robots located on the plane with ℓ_2 -norm, the makespan is at most $(1+2\sqrt{2})r$, where r is the maximum distance between the initial active robot and any asleep robot. In this paper, we prove the conjecture for the robots in the convex position and for $n \leq 7$ or $n \geq 281$, where n is the number of asleep robots (The conjecture was known to be true for $n \geq 528$ robots as shown by Bonichon et al.). Moreover, we show an upper bound of $4.63r$ for the makespan of robots in a disk of radius r in the ℓ_2 -norm, improving the best known bound of $5\sqrt{2}r \approx 7.07r$.

1 Introduction

The *freeze-tag problem* is an optimization problem concerned with waking up a swarm of asleep (inactive) robots in the shortest possible time starting with a single awake (active) robot. Consider a set of robots represented by S and $|S| = n + 1$ for $n \in \mathbb{N}$. Let p_0, \dots, p_n be the locations of the robots in a metric space, with p_0 being the location of the initial awake robot.

To activate an asleep robot, an awake robot must travel to the position of the asleep robot. As soon as an asleep robot is activated (awakened), it can assist in waking up the other asleep robots. We assume that each awake robot moves at the same speed of one unit per second while the asleep robots do not move. The *makespan (wake-up time)* is the time of the last wake-up. The freeze-tag problem has applications in group

formation, searching, and recruitment in robotics, as well as broadcasting and IP multicast problems in network design (see [2, 9] and their references).

The problem can be rephrased as follows: A *wake-up tree* of S is a binary weighted spanning tree rooted at p_0 such that the degree of p_0 is one and the length of an edge is the distance between its endpoints (see for instance Figure 1). The freeze-tag problem is to find a wake-up tree of S with the minimum (weighted) depth.

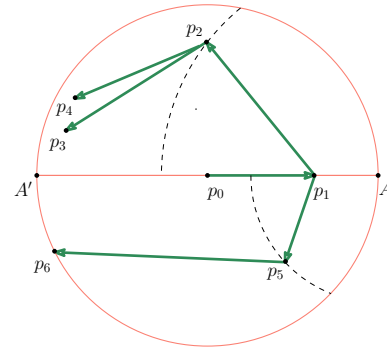


Figure 1: An example of a wake-up tree with 6 asleep robots in the Euclidean plane.

In this setup, Arkin et al. [3] give a constant approximation algorithm for the freeze-tag problem, when one asleep robot is located on each vertex. They also show that using an underlying graph metric, the problem is NP-hard. In a different paper, Arkin et al. [2] show that the freeze-tag problem, even on star metrics, is NP-complete. Moreover, they show that obtaining a $5/3$ -approximation is NP-hard for general metrics on weighted graphs. Therefore, a polynomial-time approximation scheme does not exist unless $P = NP$. In a related paper, Könemann et al. [9] consider the problem of finding a minimum diameter spanning tree with a bounded maximum degree in a complete undirected weighted graph and provide an $O(\sqrt{\log n})$ -approximation algorithm for the freeze-tag problem in the general setting.

In this paper, we consider the *geometric freeze-tag problem* for the collection of robots. In the geometric freeze-tag problem, robots are modeled as points in \mathbb{R}^d in a particular metric for some $d \in \mathbb{N}$. For $d = 3$ and ℓ_p norm, it has been shown that the geometric freeze-tag problem is NP-hard where $p \geq 1$ [6, 7, 10]. Sztainberg et al. [11] give a heuristic algorithm with a tight

*Due to the space constraint several proofs are omitted. For the proofs see the full version of the paper.

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approximation of $\Theta(\log^{1-1/d} n)$ for the makespan of n asleep robots in d dimensional space. In particular, their greedy algorithm yields an $O(1)$ -approximation in one dimension ($d = 1$) and an $O(\sqrt{\log n})$ -approximation in two dimensions ($d = 2$). Arkin et al. in [2], for any constant $d \in \mathbb{N}$, provide a polynomial-time approximation scheme when robots are located in \mathbb{R}^d equipped with ℓ_p metric. Moreover, their algorithm runs in time $O(n \log n + 2^{\text{poly}(1/\epsilon)})$.

It is worth mentioning that Hammar et al. [8] study the online freeze-tag where each asleep robot is revealed at a specified time. Later, an optimal algorithm for the online freeze-tag problem was introduced by Burnner et al. [5].

In the geometric setting, as long as normed spaces are concerned, the positions of all the robots can be scaled and translated such that all asleep robots are in a unit disk, and the initial active robot is at the origin (i.e., the active robot is at the origin and the distance between the active robot and the farthest asleep robot is a unit). Note that in this configuration, the makespan is always lower bounded by the maximal distance between the active robot and asleep robots (the radius of the unit disk). Combinatorial upper bounds for the makespan of robots in a unit disk with respect to ℓ_p norm are studied by Bonichon et al. [4]. In particular, when robots are located in the unit disk in the plane with ℓ_1 norm, they provide a tight strategy with makespan 5. They also show [4, Theorem 2] that the makespan for n asleep robots in the unit disk with one active robot at the origin is at most $3 + c/\sqrt{n}$, where c is a constant depending on the norm.

We focus on the *Euclidean freeze-tag problem on the plane*. That is, we consider robots as points in the Euclidean metric space on \mathbb{R}^2 . In this setting, the problem remains NP-hard [1], and Najafi Yazdi et al. [12] provide an algorithm with a makespan $(5 + 2\sqrt{2} + \sqrt{5})$ for the robots located in a unit square that runs in linear time. Recently, Bonichon et al. [4] proposed an algorithm with a makespan $5\sqrt{2}$. They also conjectured that the maximum makespan of robots in a unit disk of any norm is achieved when the number of robots is four. For $n = 4$, we get the worst-case whenever four asleep robots p_1, p_2, p_3 , and p_4 form a square with sides of length $\sqrt{2}$. It takes time 1 to go from the active robot p_0 in the center to p_1 , and then one robot has to wake up p_2 followed by p_3 in time $2\sqrt{2}$, and the other one wakes up p_4 . In the Euclidean freeze-tag problem, this conjecture translates to the following.

Conjecture 1 ([4]) *Let n be a positive integer. There exists a strategy to wake up n asleep robots inside a unit disk in Euclidean space starting with an active robot at the origin in time at most $1 + 2\sqrt{2}$.*

Our main contributions are the following. First we show that the Conjecture 1 holds for $n \leq 7$ or $n \geq 281$, and also when the robots are in convex position (Theorems 1, 2, and 3). Then we provide a new upper bound of 4.63 for the makespan of the Euclidean freeze-tag problem on the plane, improving upon the best-known result of $5\sqrt{2} \approx 7.07$. This also shows that the optimal upper bound for the Euclidean case is strictly less than the lower bound of 5 for the ℓ_1 norm.

The rest of this paper is organized as follows: The next section will be dedicated to preliminaries and some definitions for the geometric objects needed in the sequel. In Section 3, we discuss monotonic wake-up strategies for two simple geometric objects as a subroutine. In Section 4, as a warm-up, we prove Conjecture 1 for small values of n . Section 5 and Section 6 study the correctness of Conjecture 1 when the asleep robots are in a convex position and when the number of asleep robots is at least 281, respectively. Section 7 establishes an improved makespan of 4.63 for the Euclidean freeze-tag problem in the plane.

2 Preliminaries

For each $0 \leq i \leq n$, the *wake-up time of robot p_i* is the length of the path of p_i from p_0 in the wake-up tree. The depth of a wake-up tree indicates its makespan. Using this terminology, a closed geometric region \mathcal{R} on the plane with a specified active robot has a makespan of at most τ if for every $n \in \mathbb{N}$, there exists a wake-up tree for every configuration of n asleep robots in the region \mathcal{R} with a depth at most τ .

Many of the strategies that we will define rely on a recursive decomposition of regions \mathcal{R} into subregions. Therefore, we will define some regions that will be useful to us later on.

A *cone* of angle θ and radius r is a geometric region inside a disk of radius r between two segments with one endpoint on the center of the disk and the other endpoint on the boundary such that the angle between the two segments is θ (see Figure 2(a)). The center of the disk is referred to as the cone's apex. A cone defined using a disk of radius one is called a *unit cone*.

A (unit) *crown* of angle θ and width w is obtained from a unit cone of angle θ by subtracting a smaller cone of the same angle and radius $1 - w$ (see Figure 2(b)). Each non-trivial crown consists of 4 sides: two curved sides and two straight-line sides. We call the longer curved side of a crown the exterior side and the shorter curved side of a crown is called the interior side. For future reference, we represent the makespan of a unit crown of angle θ and width w starting with two active robots at a corner on the exterior side of the crown with

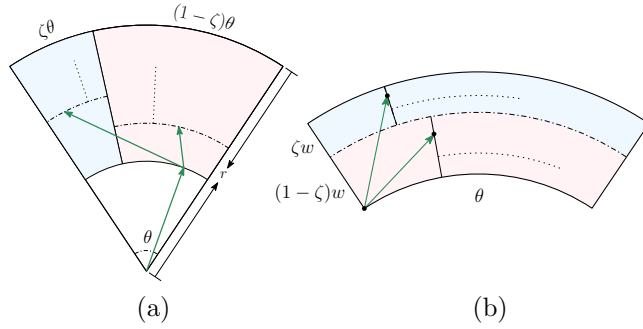


Figure 2: (a) A cone of angle θ and radius r . The monotonic wake-up strategy to solve a cone with one active robot in the apex. (b) A crown of angle θ and width w . The monotonic wake-up strategy to solve a crown with two active robots on a corner.

crown(w, θ).

A *dome* is the part of a cone between its arc and its chord (see Figure 6). The radius and angle of a dome are the same as the corresponding cone.

We define *semi-cone* to be the region enclosed between two chords of a unit disk with one common endpoint that does not contain the origin in its interior (see Figure 7(a)). We call the common endpoint of two chords the apex of the semi-cone.

Throughout this paper, ϕ stands for the golden ratio, i.e., $\phi = \frac{1+\sqrt{5}}{2}$. Note that $\phi^2 = \phi + 1$.

3 First Bounds For Geometric Shapes

In this section, we present *monotonic* wake-up strategies for two simpler geometric objects, namely the unit cone and the unit crown, as subroutines for the other algorithms discussed in the rest of the paper. We first assume that for cones (resp. crowns) robots are ordered w.r.t the distance (resp. angular distance) from the apex (resp. a corner). Given a binary wake-up tree, a strategy is said *monotonic* if for every path from the root to leaves, points are ordered w.r.t to the distance from the root.

We begin this section with a simple observation stating the triangle inequality in polar notation.

Observation 1 *Let $A = (r_a, \theta_a)$ and $B = (r_b, \theta_b)$ be two points in polar notation inside a unit disk. Then we have $\|AB\| \leq |r_a - r_b| + \max(r_a, r_b) \cdot |\theta_a - \theta_b|$. In particular, since $r_a, r_b \leq 1$, we have $\|AB\| \leq |r_a - r_b| + |\theta_a - \theta_b|$.*

In the following, $|r_a - r_b|$ and $\max(r_a, r_b) \cdot |\theta_a - \theta_b|$ are referred to as the radial distance and angular dis-

tance between A and B , respectively. As the first classical result, we present a result from [4] that establishes an upper bound for the makespan of robots positioned within a unit cone of angle θ (refer to Figure 2(a)).

Informally, the initial robot finds the closest asleep robot in the cone and the cone is partitioned into two regions, namely, subcones of angle $\zeta\phi$ and $(1 - \zeta)\phi$. Then the process is repeated similarly in each region with one active robot. From Observation 1, the length of the path p_0, p_1, \dots, p_ℓ from the initial robot toward any other robot is at most $\sum_i |r_i - r_{i-1}| + |\theta_{p_i} - \theta_{p_{i-1}}|$. The sum of the first terms is at most 1 whereas for the angular detour we have $A(\theta) \leq \max\{\theta + A(\zeta\theta), (1 - \zeta)\theta + A((1 - \zeta)\theta)\}$. If we take $\zeta = 1/2$, $A(\theta) \leq \sum_{i \geq 0} \theta/2^i < 2\theta$. In fact, taking $\zeta = 2 - \phi$, we get:

Lemma 1 ([4](**Proposition 14**)) *There exists a strategy to wake up asleep robots in a cone of angle θ and radius one starting with one awake robot at the center of the cone in time at most $1 + \phi\theta$.*

As the next geometric subroutine, similar to Lemma 1, we can construct a monotonic wake-up strategy for a unit crown using a monotonic recursive partition into sub-crowns.

Lemma 2 *There exists a strategy to wake up all of the robots in a crown of angle θ and width w starting with two awake robots at a corner in time at most $\theta + \phi w$.*

If we consider only one awake robot on the boundary, we must consider an extra time to wake up another robot, and then we can apply the result of Lemma 2.

Corollary 1 *There exists a strategy to wake up all of the robots in a crown of angle θ and width w starting with one awake robot at a corner in time at most $\theta + (1 + \phi)w$.*

Finally, with a strategy analogous to that in Lemma 2, one can wake up robots within a rectangular region.

Corollary 2 *There exists a strategy to wake up all of the robots in a rectangle of width w and height h starting with two awake robots at a corner in time at most $h + \phi w$.*

4 Configurations With Small Number of Asleep Robots

In this section, we state the correctness of Conjecture 1 for the small number of asleep robots.

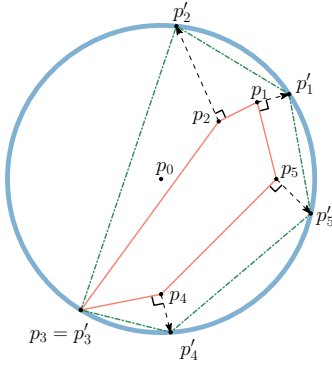


Figure 3: Projection of convex points on the disk.

Theorem 1 *Let $n \leq 7$ be a positive integer. There exists a strategy to wake up n robots in a unit disk in time less than $1 + 2\sqrt{2}$.*

The statement is trivial for $n \leq 3$. Let $n \in \{4, 5, 6, 7\}$ and p_1 be one of the robots such that the line passing through p_0 and p_1 cuts the unit disk C into two parts, each of which contains at most $\lceil (n-1)/2 \rceil$ robots (different from p_0 and p_1). Let A and A' be the intersection of this line with C , such that p_1 is on the segment p_0A .

The wake-up strategy is as follows: once the robot p_1 is awakened, the two robots are positioned at point A . Then, each robot takes care of half of the disk. In each half-disk, the robot will awaken the robot closest to A , and then the two robots will each awaken at most one more robot (see Figure 1).

5 Robots In Convex Position

In this section, we assume that the coordinates p_i are ordered in the counter-clockwise cyclic ordering. We present the following theorem:

Theorem 2 *If the point set corresponding to S is in a convex position within a disk C of radius one, the makespan of S is upper bounded by $1 + 2\sqrt{2}$.*

Let us sketch the proof of Theorem 2:

- For each $p_i \in S$, we assign the point $p'_i \in C$ being the intersection of the ray perpendicular to $p_{i-1}p_i$ emanating from p_i and going outside from the convex hull of S . This projection is such that for any pair p_i and p_j , $\|p'_i p'_j\| \geq \|p_i p_j\|$ (see Figure 3).
- If S' is a point set on the disk C , we provide a wake-up tree T' of makespan less or equal to $1 + 2\sqrt{2}$.
- The wake-up tree T on S is defined from T' . If (p'_i, p'_j) belongs to T' then (p_i, p_j) belongs to T . We

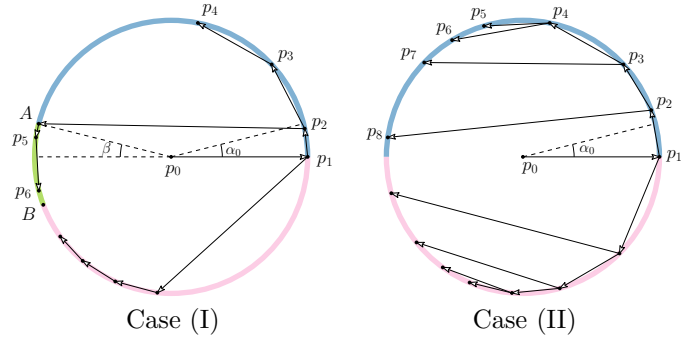


Figure 4: Robots on the disk. Case (I): The disk is partitioned into 3 arcs. Case (II): Without the existence of a small angle, the disk is partitioned into 2 half-disks. Except for the last robot p_8 , robots p_2 to p_7 are awake using a monotonic complete binary tree from p_1 .

show that the makespan of T is smaller or equal to the one of T' and thus bounded by $1 + 2\sqrt{2}$.

Let's introduce a simple strategy to wake up an arc of angle α containing k asleep robots while the first robot p_j on the arc in the counter-clockwise order is awake. In the algorithm ARC STRATEGY, p_j wakes up p_{j+1} , and any *monotonic* complete binary wake-up tree can be considered. By monotonic, we mean that each robot wakes up another robot with a larger angular position. The depth of this binary wake-up tree is $1 + \lfloor \log_2 k \rfloor$.

DISK STRATEGY takes as an input the point set S on the boundary of a unit disk and an angle α_0 . Take $\beta = \pi - 2\sqrt{2}$ and let $1 \leq i \leq n$ be an integer where the angle $\angle p_i p_0 p_{i+1}$ is the smallest. We consider two cases (see Figure 4):

- Case (I): If $\angle p_i p_0 p_{i+1} \leq \alpha_0$ then p_0 wakes up p_i . For convenience, assume that the line passing through p_0 and p_i is horizontal. The two robots located at p_i , wake up in parallel, all the robots at an angular distance at most $\pi - \beta$ from p_i ; one going in the counter-clockwise order through p_{i+1} and the other one in the clockwise order through p_{i-1} . As soon as p_{i+1} is awake, it directly goes to the position of the disk at an angular distance $\pi - \beta$ from p_i and wakes up all the remaining robots in the remaining arc of angle 2β .
- Case (II): If the previous case does not hold, take an integer $1 \leq j \leq n$ such that the angle $\angle p_{j-1} p_0 p_{j+1}$ is the smallest. Again, for convenience, assume that robots p_0 and p_j are located on a horizontal line. As in the previous case, p_0 wakes up p_j and robots on the two arcs up to the angular distance $\pi - \alpha_0$ from p_j are awake by two robots emanating from p_j using ARC STRATEGY. One robot from p_{j+1}

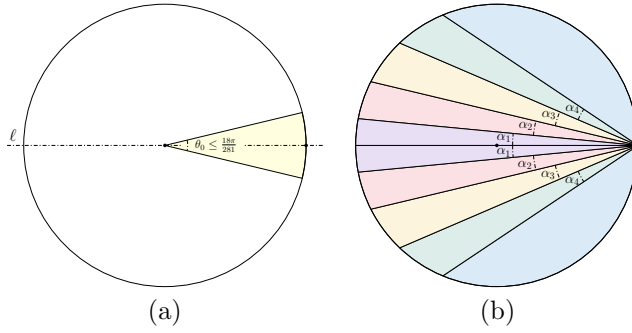


Figure 5: Illustration of proof of Theorem 3. (a) stage one. (b) stage two: split into 8 semi-cones and 2 domes.

(resp. p_{j-1}) directly goes to the last robot on the arc $(\pi - \alpha_0, \pi)$ (resp. $(\alpha_0 - \pi, -\pi)$). Then the rest of the robots on the arcs of angle $\pi - \alpha_0$ are awake.

To conclude, we show that for $\alpha_0 = \pi/11$, the DISK STRATEGY has a makespan less than or equal to $1 + 2\sqrt{2}$.

6 Strategies For Large Number of Robots

By examining the constants closely, [4, Theorem 2] implies that Conjecture 1 is correct when n , the number of asleep robots, is at least 528. In line with [4, Theorem 2], we prove that Conjecture 1 holds when the number of asleep robots in the unit disk is at least 281. To get this lower bound on the number of robots, we need to introduce a wake-up strategy for the domes and semi-cones.

The sketch of our strategy is the following: (1) Since $n \geq 281$, there is a cone of the angle at most $\frac{18\pi}{281}$ such that it contains at least 9 asleep robots. We first recruit a team of at least 10 robots (including the initial awake robot) in the cone ending at the middle of the arc of the cone Figure 5(a) (2) we partition the disk into 8 semi-cones with a specific sequence of angles and 2 domes. Each robot wakes up in parallel each of the regions (see Figure 5(b)).

Lemma 3 *There exists a strategy to wake up asleep robots in a dome of angle α and radius one in a time at most $\alpha/2 + \sin(\alpha/2) + \phi(1 - \cos(\alpha/2))$ with an awake robot on the corner of the dome.*

Let a and b be the length of two chords of a semi-cone, where $a \leq b$ and α be the angle between them. To upper bound the makespan of asleep robots with one active robot on the apex, one can simply enclose a semi-cone with a larger cone and apply the Lemma 1 to obtain a makespan of $b + b\alpha\phi$ (see Figure 7 (a)). In

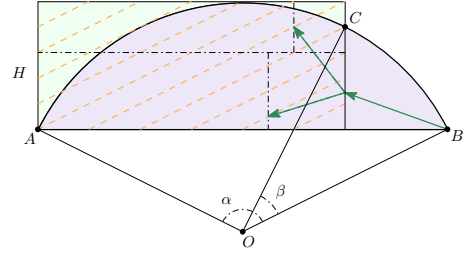


Figure 6: A dome of angle α . Illustration of proof of Lemma 3. Solving dome(α) using an enclosing rectangle.

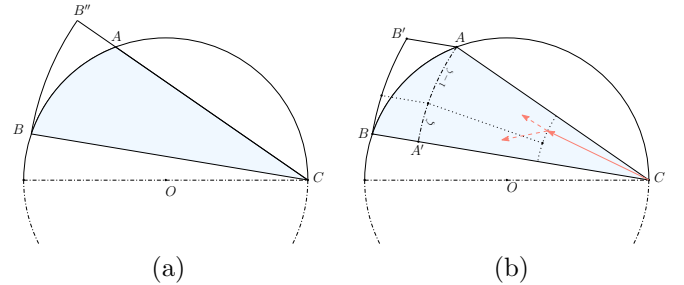


Figure 7: Illustration of proof of Lemma 4. (a) a semi-cone. (b) bounding a semi-cone with a simpler geometric object.

the following lemma, we state an upper bound $b + a\alpha\phi$ when $\alpha < \frac{\pi}{4}$.

Lemma 4 *There exists a strategy to wake up robots in a semi-cone of angle $\alpha < \frac{\pi}{4}$ and side lengths a and b where $a < b$, in time at most $b + a\alpha\phi$, where the initial active robot starts at the apex of the semi-cone.*

By having an upper bound for the makespan of a dome and semi-cone, we are ready to improve the lower bound on the number of robots needed to ensure a makespan of $1 + 2\sqrt{2}$.

Theorem 3 *Let $n \geq 281$ be an integer. There exists a strategy to wake up n robots in a unit disk in time less than $1 + 2\sqrt{2}$.*

7 A Better Upper-bound On The Wake-Up Time

In this section, we present an improved approximation on the makespan of $n \in \mathbb{N}$ asleep robots located in a unit disk on Euclidean plane. By a careful study of the first step in the analysis of Lemma 2 one can propose the following improvement on the monotonic strategy mentioned for the unit crown of width w and angle θ .

Lemma 5 *There exists a strategy to wake up all of the robots in a crown of angle θ and width w starting with two awake robots at a corner on the exterior side of the crown in time at most $\theta + \left(\frac{\phi^4}{\phi^3 + \theta}\right)w$.*

Similar to Corollary 1, one can obtain the following improved corollary for the makespan of robots starting with one active robot at a corner of the interior side of the crown.

Let q be the first asleep robot in an angular sweep starting from the active robot p_0 . If p_0 first awakes q and then the two robots directly join at the exterior side of the crown, we can then apply Lemma 1 to the rest of the crown and obtain the following corollary:

Corollary 3 *There exists a strategy to wake up all of the robots in a crown of angle $\theta \leq \pi$ and width w starting with one awake robot at a corner on the interior side of the crown in time at most $\text{crown}(w, \theta) \leq \theta + \left(1 + \frac{\phi^4}{\phi^3 + \theta}\right)w$.*

In the rest of this section, assume that the asleep robots p_1, p_2, \dots, p_n , are in sorted order based on their distance from p_0 . That is, if d_i is the distance of p_i from p_0 , for each $1 \leq i \leq n$, then we have $d_1 \leq d_2 \leq d_3 \leq \dots \leq d_n$. We introduce two basic strategies to wake up the robots in set S , and by mixing these two strategies, we get an upper bound for the makespan of n robots located in a unit disk in the plane.

Strategy One

The first awake robot at the origin, p_0 , travels to the closest robot, p_1 , at a distance of d_1 , and after activating p_1 , both of the robots, p_0 and p_1 , travel back to the origin at a total time of $2d_1$. Recall that the point at a distance of d_2 from the origin is p_2 . Let ℓ be the line that passes through the origin and p_2 . Next, p_0 and p_1 follow different paths. p_0 travels a distance of d_2 to activate p_2 . Then, p_0 and p_2 split the remaining region into two equal crowns of angle $\frac{2\pi}{3}$ and width $1 - d_2$ (See Figure 8). Simultaneously, p_1 uses a strategy as in the proof of Corollary 3 to wake up all the robots within the crown of angle $\frac{2\pi}{3}$ and width $1 - d_2$ with the bisector ℓ . Using Corollary 3, each of p_0 , p_1 and p_2 wake up their designated crown in time of at most $\frac{2\pi}{3} + \left(1 + \frac{\phi^4}{\phi^3 + \frac{2\pi}{3}}\right)(1 - d_2)$.

Note that $\frac{2\pi}{3} + \left(1 + \frac{\phi^4}{\phi^3 + \frac{2\pi}{3}}\right)(1 - d_2)$ is decreasing as function of d_2 . Since $d_1 \leq d_2$, the total makespan of this strategy, $T_1(d_1)$, as a function of d_1 , is upper bounded by:

$$T_1(d_1) \leq 2d_1 + d_2 + \text{crown}(1 - d_2, 2\pi/3)$$

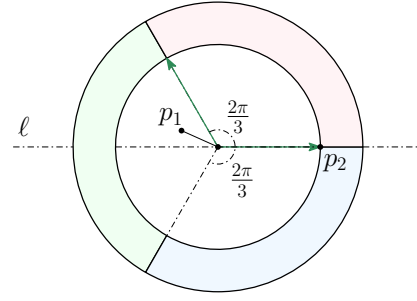


Figure 8: Strategy one. Three crowns of angle $\frac{2\pi}{3}$ and width $1 - d_1$.

$$\begin{aligned} &\leq 1 + 2d_1 + \frac{2\pi}{3} + \left(\frac{\phi^4}{\phi^3 + \frac{2\pi}{3}}\right)(1 - d_2) \\ &\leq 1 + 2d_1 + \frac{2\pi}{3} + \left(\frac{\phi^4}{\phi^3 + \frac{2\pi}{3}}\right)(1 - d_1). \end{aligned}$$

Strategy Two

Note that strategy one is *good* when d_1 is small. We use a simpler idea for the case when d_1 is large, i.e., the robots are close to the boundary of the unit disk. In this strategy, the first awake robot, p_0 , travels to the closest point p_1 at a distance of d_1 . Note that after activating p_1 , the disk of radius d_1 centered at the origin has no robots to be activated. Next, p_0 and p_1 split the remaining region into two crowns of angle π and width $1 - d_1$. Therefore, using Corollary 3, the total makespan of the second strategy, $T_2(d_1)$, as a function of d_1 , is upper bounded by:

$$T_2(d_1) \leq d_1 + \text{crown}(1 - d_1, \pi) \leq 1 + \pi + \left(\frac{\phi^4}{\phi^3 + \pi}\right)(1 - d_1).$$

Best of Two Worlds

By analyzing the best makespan of these 2 strategies, we obtain the following result:

Theorem 4 *Let n be a non-negative integer. There exists a strategy to wake up n robots within a unit disk starting with an awake robot in the center in time less than 4.6211. The construction of such a wake-up tree can be done in linear time.*

It is worth mentioning that a wake-up tree of depth at most 4.6211 can be done in linear time. Whenever n is large, the construction of such a wake-up tree can be done in linear time using LINEAR-SPLIT-STRATEGY using a partition of the disk into cones and applying

linear time for every cone (cf. Appendix B of [4]). These technical constructions are based on binary heaps and do not require any ordering.

Acknowledgement

This research was conducted at LaBRI, University of Bordeaux, during the fall of 2023, while the fourth author was visiting the other authors. The fourth author sincerely appreciates the warm hospitality and the academically enriching atmosphere at LaBRI. Furthermore, gratitude is extended to the University of Ottawa for supporting this visit through the PhD mobility program in France.

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