


Isometric-Universal Graphs for Trees

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Abstract

We consider the problem of finding the smallest graph that contains two input trees each with at most n vertices preserving their distances. In other words, we look for an isometric-universal graph with the minimum number of vertices for two given trees. We prove that this problem can be solved in time $O(n^{5/2} \log n)$. We extend this result to forests instead of trees, and propose an algorithm with running time $O(n^{7/2} \log n)$. As a key ingredient, we show that a smallest isometric-universal graph of two trees essentially is a tree. Furthermore, we prove that these results cannot be extended. Firstly, we show that deciding whether there exists an isometric-universal graph with t vertices for three forests is NP-complete. Secondly, we show that any smallest isometric-universal graph cannot be a tree for some families of three trees. This latter result has implications for greedy strategies solving the smallest isometric-universal graph problem.

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1 Introduction

Trees are one of the most natural and basic graph classes that can be studied. Studying problems on trees is thus natural, and is at the crux of the intersection between Graph Theory and Computer Science. Even for a general problem such as “finding a graph that contains two input trees”, there are a multitude of variants. To cite few of them, there is the case where the final graph is constrained to be a tree, where “containing” can be either as a subgraph or as a topological embedding [17]. There is also the more general case where the input is two graphs instead of two trees [1, 2, 8], or where the input is composed of more than two trees contained as minors [5, 16].

In this paper, we consider the problem of constructing a graph containing as isometric subgraphs two given trees (or forests) with the aim of minimizing its number of vertices. An *isometric subgraph* of a graph G is a subgraph H in which the distances between all the vertices of H are preserved in G . This is equivalent to the notion of *distance-preserving embedding* used by Winkler [19], and then reworded as *distance-preservers* [6, 7, 10] with important applications to spanners and distance labeling schemes.

This basic problem can be extended in many ways. First of all, one can consider a family of graphs as input rather than two trees. This leads to the notion of *universal graphs*, a notion that has been widely studied. More precisely, a graph \mathcal{U} is *isometric-universal* for a graph family \mathcal{F} if every graph of \mathcal{F} appears as an isometric subgraph in \mathcal{U} . In this line of



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research, \mathcal{U} and \mathcal{F} are often constrained. For instance, if \mathcal{U} and \mathcal{F} are limited to caterpillars¹, Chung, Graham, and Shearer [9] proved that n -vertex caterpillars have an isometric-universal caterpillar with $O(n^2/\log n)$ vertices, which is tight. And, for the family of all n -vertex graphs, Esperet, Gavioille, and Groenland [12] showed that there exists an isometric-universal graph with at most $3^{n+O(\log^2 n)}$ vertices, and with a lower bound of $2^{(n-1)/2}$.

The notion of isometric subgraph can also be extended to k -isometric subgraphs as follows. A subgraph H of G is called k -isometric if the distances between all the vertices of H at distance at most k in G are preserved in H . This definition generalizes several classical notions of graph containment. Indeed, a subgraph is simply a 0-isometric subgraph, an induced subgraph is a 1-isometric subgraph, and a distance-preserving embedding or isometric subgraph is an ∞ -isometric subgraph. This leads to the notion of *k -isometric-universal graphs* for a given family of graphs. Note that sparse universal graphs ($k = 0$) and small induced-universal graphs ($k = 1$) are well-studied topics in Graph Theory, see for instance [4, 11, 13] and the references therein. Thus, the initial problem we consider in this paper corresponds to the case where $k = \infty$ and \mathcal{F} is composed of two trees, and more generally two forests.

1.1 Our contribution

On the positive side, we prove that constructing a minimum isometric-universal graph for two forests, each with at most n vertices, can be done in time $O(n^{7/2} \log n)$. On the negative side, we show that this result cannot be extended to three forests by proving that the associated decision problem is NP-complete.

More precisely, in Section 2, we demonstrate that a minimum isometric-universal graph for two trees can be constructed in time $O(n^{5/2} \log n)$ thanks to an algorithm due to Gupta and Nishimura [17]. Additionally, we show how to reduce the problem for two forests from the **Assignment Problem**, a variant of the maximum matching of maximum weight problem in bipartite graphs, with an $O(n)$ slow-down factor in time complexity. As a key ingredient, we show that two trees always admit a minimum isometric-universal graph that is a tree. We believe that this nice property for two trees is an important technical contribution of the paper.

On the other hand, in Section 3, we show that these positive results cannot be extended in at least two ways. First, we show that deciding whether there exists an isometric-universal graph with t vertices for three forests is NP-complete. This is achieved by a technical reduction from the **3D-Matching Problem**. Second, we exhibit an infinite family of triples of trees having no minimum isometric-universal graph that is a tree. Implications to approximation algorithms are discussed in Section 3.2. In particular, we demonstrate that no greedy strategy can build a minimum isometric-universal graph, even with access to an optimal oracle for a pair of graphs. Finally, we show that the property used for two trees cannot be extended to minimum k -isometric-universal graphs if $k < (n - 8)/3$.

1.2 Preliminaries and definitions

We consider finite and undirected graphs with no loops and no multi-edges. The *union graph* of two graphs X, Y , denoted by $X \cup Y$, is defined by $V(X \cup Y) = V(X) \cup V(Y)$ and $E(X \cup Y) = E(X) \cup E(Y)$. Similarly, the *intersection graph* of X, Y , denoted by $X \cap Y$, is defined by $V(X \cap Y) = V(X) \cap V(Y)$ and $E(X \cap Y) = E(X) \cap E(Y)$. The disjoint union is denoted by $X \sqcup Y$.

¹ A tree whose vertices of degree at least two induce a path.

Given a graph G , and two vertices u, v of G , we denote by $\text{dist}_G(u, v)$ the minimum number of edges of a path connecting u to v in G , if it exists, and we set $\text{dist}_G(u, v) = \infty$ otherwise, i.e., if u and v are in different connected components of G .

A subgraph H of G is k -isometric if, for all vertices u, v of H , $\text{dist}_H(u, v) = \text{dist}_G(u, v)$ if $\text{dist}_G(u, v) \leq k$. Note that, in that case, if $\text{dist}_H(u, v) > k$, then $\text{dist}_G(u, v) > k$. In particular, since H is a subgraph of G , if $\text{dist}_H(u, v) = k + 1$, then $\text{dist}_G(u, v) = k + 1$.

A k -isometric-universal graph for a given graph family \mathcal{F} is a graph \mathcal{U} such that every graph G of \mathcal{F} is isomorphic to a k -isometric subgraph of \mathcal{U} . Moreover, \mathcal{U} is called *minimum* if it has the smallest possible number of vertices. \mathcal{U} is *minimal* if no proper subgraph of \mathcal{U} is a k -isometric-universal graph for \mathcal{F} . Note that if \mathcal{U} is minimal, then, for every k -isometric embedding φ from the graphs of \mathcal{F} to \mathcal{U} , we have $\mathcal{U} = \bigcup_{G \in \mathcal{F}} \varphi(G)$. Whenever $k = \infty$, we rather use the term *isometric-universal graph* as introduced by Esperet et al. [12].

2 Universal Graph for Two Forests

Let us first remark that computing the minimum isometric-universal graph is hard in general, even for two graphs. More precisely, given two graphs G, H , and some number t , the k -Isometric-Universal Graph for Two Graphs Problem asks for finding a k -isometric-universal graph with t vertices for $\{G, H\}$, that is a t -vertex graph \mathcal{U} containing G and H as k -isometric subgraphs.

Indeed, the problem is NP-complete for each $k \in \mathbb{N}$ and for two graphs with the same number of vertices. For $k = 0$, this is simple reduction² from **Hamiltonian Path**, and for $k > 0$, this is a reduction from **Clique** of size p . Indeed, for a graph G with n vertices, set H to a path (if $k = 0$) or to a clique (for $k > 0$) with n vertices, and ask whether there exists a k -isometric-universal graph for the family $\{G, H\}$, with respectively $t = n$ vertices for $k = 0$, or with $t = 2n - p$ vertices for $k > 0$.

We will consider the variant where the pair $\{G, H\}$ is composed of two forests. The problem is known to be NP-complete for $k = 0$, even if one of the two forests is a tree [14, Theorem 4.6, p. 105]. As shown by Gavaille and Jacques [15], this NP-completeness result can be extended to two trees of the same size for $k = 1$. Nevertheless, as we will see in this section, the problem becomes polynomial for two forests if $k = \infty$.

2.1 Connected components reduction

We start with the following result, that will allow us to reduce the problem of computing the minimum isometric-universal graph for a pair of graphs (possibly non-connected) to the case of their connected components. The reduction is based on a polynomial solution of the **Assignment Problem** whose goal is to optimally assign agents to individual tasks. For the purpose of the next statement, we say that a function f is *superlinear* if $f(n)/n$ is non-decreasing. Typically, polynomials n^{1+c} and exponentials 2^{cn} for $c > 0$ are superlinear functions.

► **Theorem 1.** *Let G, H be two graphs with at most n vertices, composed respectively of s and r connected components. Let $f(m)$ be any superlinear function bounding above the time complexity of computing a minimum isometric-universal graph for any pair of graphs*

² It is easy to check that the problem is in NP, the certificate being composed of the universal graph and of the embedding from each graph to the universal graph, both can be described polynomially in the size of G, H .

with at most m vertices composed of a component of G and of H . Then, assuming $s \leq r$, a minimum isometric-universal graph for $\{G, H\}$ can be computed in time

$$O\left(r \min\{f(2n), sf(n)\} + s \min\{r\sqrt{s} \log n, rs + s \log \log s\}\right).$$

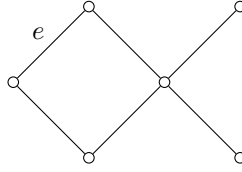
The proof is available in the full version [3].

2.2 The case of two trees

To compute the minimum isometric-universal graph of two forests in polynomial time, we will combine Theorem 1 and the next result about two trees.

► **Theorem 2.** *Every minimum and minimal isometric-universal graph for two trees is a tree.*

Before presenting its technical proof, we remark that both properties, *minimum* and *minimal*, are required. Indeed, the disjoint union of any pair of trees is isometric-universal for this pair, is minimal but not minimum (as implied by Theorem 2, the minimum one must be connected). Moreover, the graph composed of two pendant vertices attached to some vertex of a C_4 (see Figure 1) is a minimum isometric-universal graph for the family $\{K_{1,4}, K_{1,3}^+\}$ composed of two 5-vertex trees, where $K_{1,p}^+$ denotes here the graph $K_{1,p}$ where one edge has been subdivided into two edges. However, it is not minimal: it contains $K_{1,4}^+$ as proper subgraph which is isometric-universal for $\{K_{1,4}, K_{1,3}^+\}$. This example can be generalized to two trees with n vertices, for every $n \geq 5$.



■ **Figure 1** A minimum isometric-universal graph for the 5-vertex tree family $\{K_{1,4}, K_{1,3}^+\}$. From Theorem 2 it is not minimal (the edge e can be removed).

Proof of Theorem 2. Let \mathcal{U} be any minimum and minimal isometric-universal graph for $\mathcal{F} = \{T_1, T_2\}$ composed of two trees. We consider a given isometric embedding of T_1 and T_2 in \mathcal{U} , and denote by A_1 and A_2 their corresponding embedded subgraphs in \mathcal{U} . Since \mathcal{U} is minimal, $\mathcal{U} = A_1 \cup A_2$. So, each vertex or edge of \mathcal{U} , belongs either to $A_1 \setminus A_2$, $A_2 \setminus A_1$, or to $A_1 \cap A_2$.

For convenience, let us denote by $d(x, y) = \text{dist}_{\mathcal{U}}(x, y)$, and by $|P|$ the *length* of path P , i.e., its number of edges. We will often use the property that every path P of A_i , for some $i \in \{1, 2\}$, satisfies $|P| = \text{dist}_{A_i}(u, v) = d(u, v)$, where u, v are the extremities of P . This is because every path of a tree is a shortest path in this tree, and because A_i is an isometric subgraph of \mathcal{U} .

\mathcal{U} must be connected, otherwise we could merge two connected components of \mathcal{U} by a vertex, and this without modifying any distance in A_1 or A_2 which are finite. This would contradicting the fact that \mathcal{U} is minimum. If \mathcal{U} is a tree, we are done. It remains to show that if \mathcal{U} has a cycle, then \mathcal{U} is not minimum. Consider the shortest cycle C in \mathcal{U} . Let us show that:

▷ Claim 3. The edges of C can be partitioned into two paths P_1 and P_2 such that (see Figure 2 on the left):

- $P_1 \cap P_2 = \{u, v\}$ with $u, v \in A_1 \cap A_2$
- $P_1 \setminus P_2 \subset A_1 \setminus A_2$
- $P_2 \setminus P_1 \subset A_2 \setminus A_1$

Proof. Consider the connected components of $A_1 \cap A_2$. Obviously, such components are trees. In particular, $C \subset A_1 \cap A_2$ is impossible. It is also clear that C intersects at least two such components. Indeed, first, if it intersects no components, then C is wholly contained in A_1 or in A_2 , contradicting they are trees. Secondly, if it intersects only one, then C is wholly contained in A_1 or in A_2 as well, since the part of C outside the component exists and belongs to one tree, and the part inside the component belongs to both trees.

Define u and v as any two vertices of C that belong to two different components of $A_1 \cap A_2$, and such that $\text{dist}_C(u, v)$ is minimum. Let P_1 be a path of C realizing this distance. By construction, the only vertices of P_1 that are in $A_1 \cap A_2$ are precisely its extremities u, v . Indeed, if $P_1 \setminus \{u, v\}$ contains a vertex w in $A_1 \cap A_2$, then both $\text{dist}_C(u, w)$ and $\text{dist}_C(v, w)$ would be less than $\text{dist}_C(u, v)$, which is not possible by the choice of $u, v \in C$. Thus, $P_1 \setminus \{u, v\}$ contains only vertices and edges of a same tree A_i , because if two consecutive edges belong to different trees, their common vertex must be in the intersection. W.l.o.g., assume that $P_1 \subset A_1$. Note that we have $\text{dist}_C(u, v) = d_{A_1}(u, v) = d(u, v) = |P_1|$ because P_1 is a path of A_1 . We set P_2 to be the path of C induced by the edges of C not in P_1 .

Clearly, the first two points hold: $P_1 \cap P_2 = \{u, v\}$ and $u, v \in A_1 \cap A_2$, and we have seen that $P_1 \setminus \{u, v\} \subset A_1 \setminus A_2$, and thus $P_1 \setminus P_2 \subset A_1 \setminus A_2$.

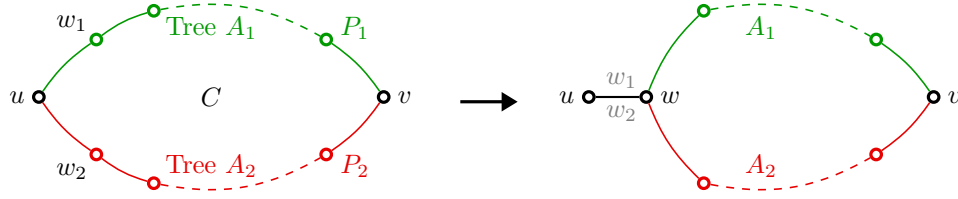
For the third point, by way of contradiction, assume that $P_2 \setminus \{u, v\}$ contains a vertex of $w \in A_1 \cap A_2$. We have $|P_1| = \text{dist}_{A_1}(u, v) = d(u, v)$. By the choice of u and v on C , we must have $|P_1| + |P_2| \geq 2d(u, v)$, i.e., $|P_2| \geq d(u, v)$. On the other hand, let us consider the path P between u and v in A_2 . We have $|P| = \text{dist}_{A_2}(u, v) = d(u, v)$. So, $P_1 \cup P$ creates a cycle of length at most $2d(u, v)$. By minimality of C , this implies that $|P_2| = d(u, v)$. But, if $w \in P_2 \setminus \{u, v\}$, then both $\text{dist}_C(u, w)$ and $\text{dist}_C(v, w)$ would be less than $|P_2| = d(u, v) = \text{dist}_C(u, v)$: a contradiction with the choice of $u, v \in A_1 \cap A_2$ that minimizes the distance on C . So, $P_2 \setminus P_1$ has no vertex in $A_1 \cap A_2$. Thus, $P_2 \setminus P_1$ contains only vertices and edges of a same tree A_i . Clearly, $P_2 \setminus P_1 \subset A_1$ is not possible, since otherwise C would be wholly contained in A_1 . So, $P_2 \setminus P_1 \subset A_2 \setminus A_1$ as required. ◁

Let u, v, P_1, P_2 be as in Claim 3. Note that $d(u, v) > 1$. Indeed, we have seen that the size of C is $|P_1| + |P_2| = 2d(u, v)$, and $d(u, v) = 1$ is not possible since C is a cycle. Then, we can define w_i to be the neighbor of u in P_i , for each $i \in \{1, 2\}$. See Figure 2 on the left. And, consider the graph \mathcal{U}' obtained from \mathcal{U} by identifying w_1 and w_2 into a new vertex w , and by removing double edges created. We call this operation on \mathcal{U} a *contraction*, since it corresponds to a classical edge-contraction if w_1 and w_2 are adjacent. See Figure 2.

We will show that \mathcal{U}' is still isometric-universal for \mathcal{F} , contradicting the fact that \mathcal{U} has the minimum number of vertices. This will prove that C does not exist in \mathcal{U} , and so \mathcal{U} is a tree.

Let us show that A_1 and A_2 are isometric subgraphs of \mathcal{U}' . We have to prove that the contraction of w_1 and w_2 does not change any distance between any pair of vertices that belong to a same tree A_i , for each $i \in \{1, 2\}$.

First we remark that A_i is a subgraph of \mathcal{U}' . Indeed, w_1 and w_2 belong to different trees, and thus the edge $u - w_i$ of A_i in \mathcal{U} , after contraction, is in $A_1 \cap A_2$ in \mathcal{U}' . So, the distances between vertices taken in A_i can only decrease in \mathcal{U}' .



■ **Figure 2** Cycle $C = P_1 \cup P_2$. Contracting w_1 and w_2 in \mathcal{U} . Green vertices and edges are in $A_1 \setminus A_2$, red vertices and edges are in $A_2 \setminus A_1$, and black vertices and edges are in $A_1 \cap A_2$.

We proceed by way of contradiction. Assume there are two vertices x and y at distance d in \mathcal{U} that both belong to A_i , for some i , and such that $\text{dist}_{\mathcal{U}'}(x, y) < d$. We consider only the case $A_i = A_1$, the case $A_i = A_2$ being symmetric.

Let us show that:

▷ **Claim 4.** There exists a path P between x and y in \mathcal{U} of length $|P| \leq d + 1$ that passes through the path $w_1 - u - w_2$ or the path $w_2 - u - w_1$.

Proof. For every path Q in \mathcal{U} , we denote by \hat{Q} , its *contraction*, i.e., the path obtained by contracting all the edges of the subpath of Q between w_1 and w_2 . If w_1 or w_2 is not in Q , we set $\hat{Q} = Q$. Note that \hat{Q} is a path of \mathcal{U}' .

Let P' be a shortest path in \mathcal{U}' between x and y . By hypothesis on x, y , $|\hat{P}'| < d$. Let us show that there exists a path P in \mathcal{U} such that $P' = \hat{P}$ and $|P| \leq d + 1$. If P' does not contain w , then $P = P' = \hat{P}$. Otherwise, $w \in P'$. There are two cases: (1) if P' contain uw , then we can set P as the path P' where w is replaced w_1 or w_2 (depending on the other edge incident to w in P'); (2) if P' does not contain uw , then since P' is a shortest path, it cannot contain u . We can set P as the path P' where w is replaced by the path $w_1 - u - w_2$ or $w_2 - u - w_1$. We check that in both cases, $\hat{P} = P'$, and $|P| \leq |\hat{Q}| + 2 < d + 2$, that implies $|P| \leq d + 1$. ◁

Let P be a path as in Claim 4. Note that P is a witness for $\text{dist}_{\mathcal{U}'}(x, y) < d$, i.e., a path connecting x to y and such that after contracting w_1 and w_2 , its length decreases by 2 in \mathcal{U}' , and becomes less than d . W.l.o.g. we will assume that P visits $w_1 - u - w_2$ in this order when going from x to y . If not, we can exchange the role of x and y .

Now, let us define the vertices a, b, z as follows (see Figure 3):

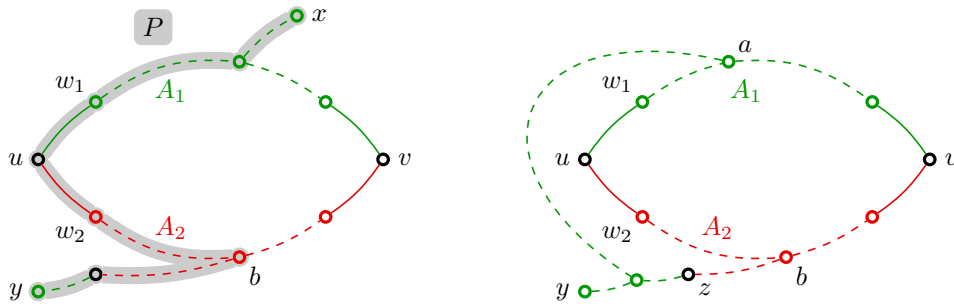
- Vertex b is the last vertex of P , when going from u to y , that belongs to P_2 . Note that b is between w_2 and y on P , possibly with $b = w_2$. See Figure 3 on the left.
- Vertex z is the first vertex of P , when going from b to y , that belongs to A_1 , possibly with $z = y$. Note that $z \neq b$. See Figure 3 on the right.
- Vertex a is the last vertex of the path between u to z in A_1 , when going from u to z , that belongs to P_1 , possibly with $a \in \{u, w_1, v\}$. See Figure 3 on the right.

Since P passes successively through vertices x, w_1, u, w_2, b, z, y , we have:

$$d(x, w_1) + d(w_1, u) + d(u, w_2) + d(w_2, b) + d(b, z) + d(z, y) \leq |P|.$$

The quantity $d(w_1, u) + d(u, w_2)$ is equal to 2, since w_1, u, w_2 are consecutive in P . On the other hand, $|P| \leq d + 1$ by definition of P . By the triangle inequality between x and y , $d = d(x, y) \leq d(x, w_1) + d(w_1, a) + d(a, z) + d(z, y)$. Therefore:

$$\begin{aligned} d(x, w_1) + 2 + d(w_2, b) + d(b, z) + d(z, y) &\leq d(x, w_1) + d(w_1, a) + d(a, z) + d(z, y) + 1 \\ \Rightarrow 1 + d(w_2, b) + d(b, z) &\leq d(w_1, a) + d(a, z) \end{aligned} \quad (1)$$



■ **Figure 3** Path P between x and y , and vertices a, b, z .

In fact, we will show that:

▷ **Claim 5.** $d(w_1, a) + d(a, z) \leq d(w_2, b) + d(b, z)$.

Before presenting its proof, observe that the inequality of Claim 5 contradicts Equation (1), and the fact that some distance between two vertices of the same tree, namely x and y , is shortened in \mathcal{U}' . So \mathcal{U}' is universal-isometric for $\{A_1, A_2\}$, and thus \mathcal{U} cannot be minimum, and thus does not contain any cycle.

Proof. We will actually prove a stronger statement which is:

$$d(w_1, a) = d(w_2, b) \quad \text{and} \quad d(a, z) = d(b, z) .$$

We first remark that vertex $z \in A_1 \cap A_2$. Indeed, we have $z \in A_1$. If $z \notin A_2$, then the edge $z - z'$ on the path P from z to b is not in A_2 . So, $z - z' \in A_1$, which implies $z' \in A_1$: a contradiction with the fact that z is the first vertex on P from b to y that is in P_1 . So $z \in A_1 \cap A_2$.

As a preliminary step, let us show that $d(u, a) = d(u, b)$, cf. Equation (5) hereafter. Recall that $u, v, z \in A_1 \cap A_2$, and that distances in A_i coincide with distances in \mathcal{U} .

By definition, a is on the path from u to z in A_1 . Thus

$$\begin{aligned} \text{dist}_{A_1}(u, z) &= \text{dist}_{A_1}(u, a) + \text{dist}_{A_1}(a, z) \\ \Rightarrow d(u, z) &= d(u, a) + d(a, z) \end{aligned}$$

Similarly for b , which is on the path in A_2 from u to z , we have $d(u, z) = d(u, b) + d(b, z)$. Combining these two equations, we get:

$$d(u, a) + d(a, z) = d(u, b) + d(b, z) \tag{2}$$

We use the following fact:

► **Fact.** *For any three vertices in a tree, the three paths between them have exactly one vertex in common.*

In particular, whenever applied to u, v and z in A_1 , pairwise connected by the three tree paths P_{uv}, P_{vz}, P_{uz} in A_1 , the fact implies the last vertex of P_{uz} that is on P_{uv} is also the last vertex of P_{vz} that is on P_{uv} . From this, we deduce that a is on the path from v to z in A_1 . Thus $d(v, z) = d(v, a) + d(a, z)$. With the same argument applied to the same

vertices but in A_2 , we deduce that b is on the shortest path in A_2 from v to z , we have $d(v, z) = d(v, b) + d(b, z)$. Therefore:

$$d(v, a) + d(a, z) = d(v, b) + d(b, z) \quad (3)$$

By subtracting Equation (3) from Equation (2), we have:

$$\begin{aligned} (d(u, a) + d(a, z)) - (d(v, a) + d(a, z)) &= (d(u, b) + d(b, z)) - (d(v, b) + d(b, z)) \\ \Rightarrow d(u, a) - d(v, a) &= d(u, b) - d(v, b) \end{aligned} \quad (4)$$

Note that P_1 and P_2 are paths of A_1 and A_2 respectively, thus they are shortest paths in \mathcal{U} . Since a and b are on P_1 and P_2 respectively, we have $d(u, v) = d(u, a) + d(a, v) = d(u, b) + d(b, v)$. And by adding to Equation (4), we get:

$$d(a, u) = d(b, u) \quad (5)$$

as claimed.

We now show that:

$$d(a, w_1) = d(b, w_2) \quad (6)$$

Indeed, either $d(a, u) = d(b, u) = 0$, i.e., $a = u = b$, and thus $d(a, w_1) = d(u, w_1) = 1 = d(u, w_2) = d(b, w_2)$. Or, $d(a, u) = d(b, u) > 0$, and thus w_1 is on the path from u to a on P_1 , and w_2 is on the path from u to b on P_2 . This implies $d(u, w_1) + d(w_1, a) = d(u, w_2) + d(w_2, b)$, and thus $d(w_1, a) = d(w_2, b)$.

Furthermore, applying Equation (5) on Equation (2) gives that:

$$d(a, z) = d(b, z) \quad (7)$$

Equation (6) and Equation (7) completes the proof of Claim 5. \triangleleft

This completes the proof of Theorem 2. \blacktriangleleft

This structural result allows us to predict that every minimum isometric-universal graph of two trees $\{T_1, T_2\}$ either is a tree, or contains an isometric-universal graph for $\{T_1, T_2\}$ as a subgraph (only by removing edges). However, as the proof of Theorem 6 will explain, if we can compute a tree that contains T_1 and T_2 as subgraphs, then we can compute a minimum isometric-universal graph for $\{T_1, T_2\}$. More precisely:

► **Theorem 6.** *A minimum isometric-universal graph for a pair of trees with at most n vertices each can be computed in time $O(n^{5/2} \log n)$.*

Proof. Let T_1, T_2 be two trees with at most n vertices. From Theorem 2, there always exists a minimum isometric-universal graph of two trees that is a tree itself. Thus, any algorithm that computes a minimum isometric-universal tree for $\{T_1, T_2\}$ computes a minimum isometric-universal graph for $\{T_1, T_2\}$ too.

In a tree there exists only one path connecting any pair of vertices. Thus, every tree that contains T_1 and T_2 as subgraphs (such a tree is called a *supertree* under graph isomorphism in [17]) is also an isometric-universal tree. According to Gupta and Nishimura [17, Theorem 9.6], a minimum supertree of two trees can be computed in time $O(n^{5/2} \log n)$.

From the above discussion, the minimum supertree for T_1, T_2 is a minimum isometric-universal tree and also a minimum isometric-universal graph for $\{T_1, T_2\}$, which concludes the proof. \blacktriangleleft

2.3 The case of two forests

Now we extend the previous result to forests instead of trees. The combination of Theorem 1 and Theorem 6 leads to the following result:

► **Theorem 7.** *A minimum isometric-universal graph for a pair of forests with at most n vertices each can be computed in time $O(n^{7/2} \log n)$.*

Proof. This is a direct application of Theorem 1 applied on two forests, composed respectively of s and r trees. In the complexity formula of Theorem 1, we can bound r , s , and $f(n)$ as follows:

- $s \leq r \leq n$.
- $f(n) = Cn^{5/2} \log_2 n$, for some constant $C > 0$, since, from Theorem 6, the time complexity of finding a minimum isometric-universal graph of two trees is at most $f(n)$. Note that f is superlinear.

We have:

$$\begin{aligned}
 & r \min \{f(2n), sf(n)\} + s \min \{r\sqrt{s} \log n, rs + s \log \log s\} \\
 & \leq rf(2n) + sr\sqrt{s} \log n \\
 & = O(n^{7/2} \log n + n^{5/2} \log n) \\
 & = O(n^{7/2} \log n) .
 \end{aligned}$$

Applying Theorem 1, this completes the proof. ◀

3 Universal Graph for Three Forests

3.1 Three trees and more

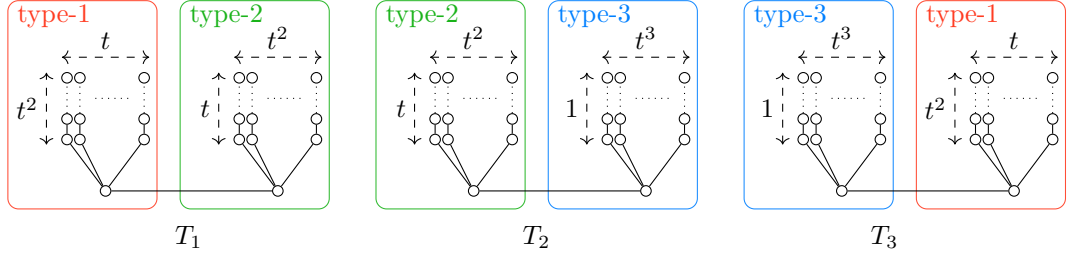
Unfortunately, Theorem 2 cannot be extended to three trees. Indeed, as we will show in Theorem 8, there are triples of trees with no minimum isometric-universal graph that is a tree. As we will see later in Section 3.2, this implies that no greedy strategy can build a minimum isometric-universal graph, even if equipped with an optimal oracle for isometric-universal graphs for a pair of graphs.

► **Theorem 8.** *For every integer $t \geq 9$, there is a family of trees $\{T_1, T_2, T_3\}$, each with $2t^3 + 2$ vertices and with pathwidth³ two, for which every isometric-universal tree is not minimum.*

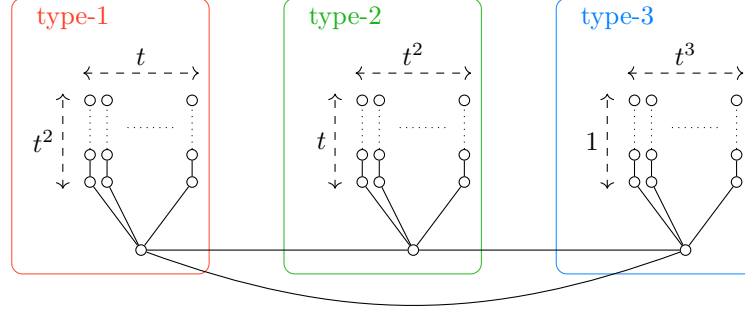
The proof is available in the full version [3]. See Figure 4 and Figure 5 for an illustration of the construction.

The smallest example provided by Theorem 8, for $t = 9$, consists of three trees of $2 \cdot 9^3 + 2 = 1460$ vertices. We suspect that this vertex amount can be lowered significantly, with perhaps more complex arguments.

³ The pathwidth of a graph G is the smallest integer p such that G is a subgraph of some interval graphs with maximum clique size $p + 1$.



■ **Figure 4** A tree family $\{T_1, T_2, T_3\}$ whose every minimum isometric-universal graph is not a tree.



■ **Figure 5** A cyclic isometric-universal graph for \mathcal{F} that is smaller than any isometric-universal tree for \mathcal{F} .

3.2 Consequences for greedy strategies

As we will see, Theorem 2 and Theorem 8 imply that there exist graph families where no greedy strategies can build a minimum isometric-universal graph. Informally, a greedy strategy consists in starting with some initial graph of a given family \mathcal{F} , and then, at each step, picking a new graph of \mathcal{F} and computing a minimum isometric-universal graph for the pair composed of the current graph and this new graph, and updates the current graph with this new isometric-universal graph.

More formally, let A be any algorithm (considered as a black-box or oracle) computing an isometric-universal graph for a pair of graphs. A greedy strategy, parametrized by A , selects the graphs of a given family \mathcal{F} in some order, say $G_1, \dots, G_{|\mathcal{F}|}$, and builds the following sequence of graphs $\mathcal{U}_1, \dots, \mathcal{U}_{|\mathcal{F}|}$:

$$\mathcal{U}_i = \begin{cases} G_1 & \text{if } i = 1 \\ A(\mathcal{U}_{i-1}, G_i) & \text{if } i > 1 \end{cases}$$

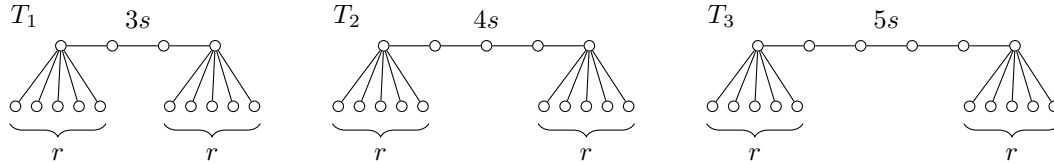
By construction, $\mathcal{U}_{|\mathcal{F}|}$ is an isometric-universal graph for \mathcal{F} , since, by induction, \mathcal{U}_i is an isometric-universal graph for $\{G_1, \dots, G_i\}$.

Unfortunately, no greedy strategies can determine the minimum isometric-universal graph for all families of trees, if the routine A constructs a minimum and minimal isometric-universal graph for any two graphs. (Recall that, for two general graphs, this is NP-hard.) This is due to the fact that, at every step i , $A(\mathcal{U}_{i-1}, G_i)$ will return a tree, since, by Theorem 2, every minimum and minimal isometric-universal graph of two trees has to be a tree. In particular, the final graph $\mathcal{U}_{|\mathcal{F}|}$ will be a tree.

Now, by Theorem 8, there is a family of trees whose minimum isometric-universal graph cannot be a tree. So, it implies that whatever the order of the trees chosen for the greedy strategy, the result cannot be a minimum isometric-universal graph.

Since by Theorem 2, $A(\mathcal{U}_{i-1}, G_i)$ must be a tree, it is legitimate to ask whether, for tree families, greedy strategies can help in computing a minimum isometric-universal *tree*, rather than a minimum isometric-universal graph. Unfortunately, this will fail as well for the following reason.

Consider the family $\mathcal{F} = \{T_1, T_2, T_3\}$, where T_i is composed of two copies of a $K_{1,r}$, whose centers are connected by a path of length $(i+2)s$, where r and $s = o(r)$ are large enough. See Figure 6 for an example.

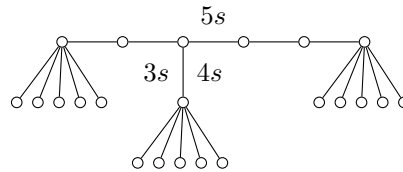


■ **Figure 6** A family $\{T_1, T_2, T_3\}$ of trees, here for $r = 5$ and $s = 1$, whose any greedy strategy designed for trees will fail.

It is not difficult to see that the minimum isometric-universal graph of $\{T_i, T_j\}$, i.e., $A(T_i, T_j)$, must contain three stars, i.e., three copies of $K_{1,r}$ intersecting only within a constant number of vertices. Indeed, assuming⁴ $i < j$, $A(T_i, T_j)$ must be composed of T_j where the center of a star of T_i belongs to the path of T_j , the other star being in common. In this universal tree, the distances between these three centers are $(2+i)s$, $(2+j)s$, and $(j-i)s$. For instance, for $i = 1$ and $j = 3$, we have the distance set $\{3s, 5s, 2s\}$. The two other distance sets being $\{3s, 4s, s\}$ (for $i, j = 1, 2$) and $\{4s, 5s, s\}$ (for $i, j = 2, 3$).

Now, by adding the third tree T_k to $A(T_i, T_j)$, by computing $A(A(T_i, T_j), T_k)$, one obtain a universal tree that must contain four stars. Indeed, the only way to get three stars would be to share both stars of T_k with two out of the three stars already in $A(T_i, T_j)$. Unfortunately, the distance $(2+k)s$ between the two star centers of T_k does not belong to the distance set $\{(2+i)s, (2+j)s, (j-i)s\}$ of $A(T_i, T_j)$.

The construction as depicted in Figure 7 shows that $\{T_1, T_2, T_3\}$ has an isometric-universal tree with only three stars, and thus with $3r + O(s)$ vertices, whereas the greedy strategy $A(A(T_i, T_j), T_k)$ gives $4r + O(s)$ vertices, which is bigger for $s = o(r)$ for instance.



■ **Figure 7** An isometric-universal tree for $\{T_1, T_2, T_3\}$, depicted in Figure 6, with only $3r + O(s)$ vertices.

To summarize the above discussion, we have shown:

► **Proposition 9.** *Let A be any algorithm computing a minimum and minimal isometric-universal graph for a pair of graphs. Then, no greedy strategies based on A can compute a minimum isometric-universal graph (or tree) for all tree families.*

⁴ As $A(T_i, T_j) = A(T_j, T_i)$, the order of the two first trees does not matter.

We note that we have analyzed greedy strategies for its *best-case* running (and not its worst-case as usual), that is for best ordering of the graphs in the family. So, these greedy strategies fail in a strong sense (i.e. whatever the ordering is).

3.3 Extending Theorem 2 to k -isometric-universal trees

Another natural extension of Theorem 2 is to ask whether it holds for k -isometric-universal graphs, for k large enough and depending of n . More formally, we may ask what is the smallest function $k^*(n)$ defined as:

|| For every $k \geq k^*(n)$, every minimum and minimal k -isometric-universal graph for a pair of trees each with at most n vertices is a tree.

Clearly, every graph \mathcal{U} containing an n -vertex tree T as an $(n-1)$ -isometric subgraph, also contains T as an isometric subgraph. Thus, by Theorem 2, $k^*(n) \leq n-1$ for all $n \geq 1$. On the other hand, as we will see hereafter in Theorem 10, there are pairs of n -vertex trees for which every minimum 1-isometric-universal graph is not a tree whenever $n \geq 11$. Thus, $k^*(n) > 1$ for all $n \geq 11$.

We establish the following lower bound for $k^*(n)$:

► **Theorem 10.** *For every $n \geq 8$, $k^*(n) > \lfloor (n-8)/3 \rfloor$. In other words, there is a pair of trees, each with at most n vertices, for which every minimum $\lfloor (n-8)/3 \rfloor$ -isometric-universal graph is not a tree.*

The proof is available in the full version [3].

Despite Theorem 10 showing that $k^*(n) \geq n/3 - O(1)$, we suspect that $k^*(n) = n/2 + o(n)$, and we leave open to determine the right asymptotic value for $k^*(n)$. On one side we believe that the above proof can be enhanced to reach $k^*(n) \geq n/2 - o(n)$. On the other hand, in the $k^*(n)$ -isometric-universal graph, it seems far-fetched to have two disjoint paths of length $> n/2$.

3.4 NP-completeness

In this section we show that Theorem 7 cannot be extended to more than two forests (unless $P = NP$). In several decision problems, setting the input from 2 to 3 costs to pass from P to NP-C (for instance 2-SAT has a polynomial solution and 3-SAT is NP-complete, same for graph 2/3-coloring). We present here the same pattern with the problem of isometric-universal graph from 2 to 3 forests. As a byproduct of our proof, the NP-completeness holds if we restrict the isometric-universal graph to be a forest. This restriction has also been claimed by Rautenbach and Werner [18, Theorem 3].

► **Theorem 11.** *Determining whether there exists an isometric-universal graph (or forest) with t vertices for three forests is NP-complete, even if the three forests have pathwidth at most two.*

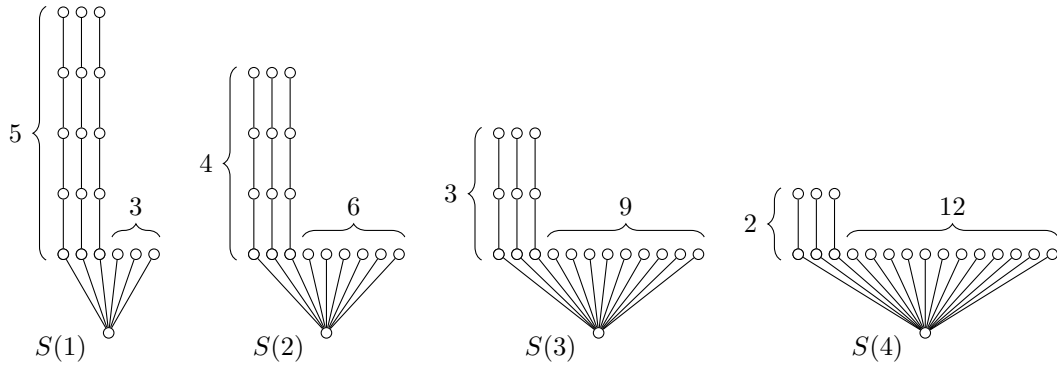
Proof. It is not difficult to see that this problem is in NP. We use a reduction from the 3D-Matching problem that is NP-complete even if each element is contained in at most three sets (see [14, Problem SP1 in Appendix A.3.1]). This restriction can be seen as determining a perfect matching in a 3-partite sub-cubic hypergraph.

For the 3D-Matching problem, we are given three disjoint sets X, Y, Z of size n , and a set of triples $T \subseteq X \times Y \times Z$. We ask whether there exists or not a subset of n triples of T that are pairwise independent. In other words, a solution for the problem, called *matching*, is a subset $M \subseteq T$ of size n such that no two triples of M agree in any coordinate.

In order to encode the 3D-Matching problem into our problem, consider a instance (X, Y, Z, T) where each element of X, Y , or Z is contained in two or three triples⁵ of T . We have $|X| = |Y| = |Z| = n$ and $|T| \in \{2n, \dots, 3n\}$, so that the instance is of size $O(n)$.

We construct a family $\mathcal{F} = \{F^X, F^Y, F^Z\}$ composed of three forests, each with n trees, one tree for each element of X, Y , and Z . More precisely, for each set $W \in \{X, Y, Z\}$, the set of connected components of F^W is denoted by $\{F(w) : w \in W\}$, where $F(w)$ is the tree associated with the element $w \in W$ of the forest F^W . Each tree $F(w)$ will be described precisely hereafter.

For every $w \in \{1, \dots, 2n\}$, let $S(w)$ be a subdivided star obtained from a $K_{1,3w+3}$ whose three of its edges have been replaced by a path of length $2n+2-w$. Let $\mathcal{S} = \{S(1), \dots, S(2n)\}$ be the collection of all such stars that are pairwise non-isomorphic. See Figure 8 for examples of such stars.



■ **Figure 8** The set $\mathcal{S} = \{S(1), \dots, S(2n)\}$ of subdivided stars for $n = 2$.

It is easy to check that:

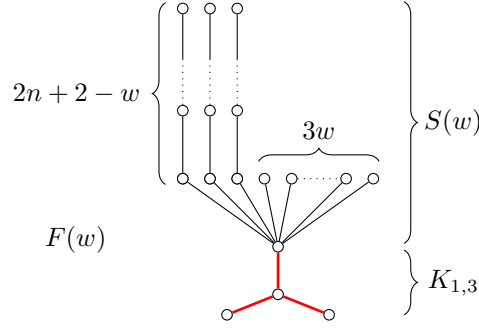
▷ **Claim 12.** Each $S(w) \in \mathcal{S}$ is a tree of pathwidth two with $3n + 1$ vertices, its center has degree $3w + 3$, $3w$ branches of length one, and three branches of length $2n + 2 - w \geq 2$.

The forests F^Y and F^Z . For convenience, we will assume that $Y = \{1, \dots, n\}$ and $Z = \{n + 1, \dots, 2n\}$, so that $Y \cup Z = \{1, \dots, 2n\}$. Actually, any one-to-one mapping from $Y \cup Z$ to $\{1, \dots, 2n\}$ is fine. For each $w \in Y \cup Z$, the tree $F(w)$ is composed of a copy of the star $S(w) \in \mathcal{S}$ where is attached a *claw*, i.e., a copy of a $K_{1,3}$. More precisely, the center of $S(w)$ is merged with a leaf of the claw. The tree $F(w)$ has $|V(S(w))| + |V(K_{1,3})| - 1 = 3n + 4$ vertices, and pathwidth two. See Figure 9.

The forest F^X . Consider any $x \in X$, and define $T(x) = \{(x, y, z) \in T\}$ to be the set of triples of T containing element x . From the restriction of the input, $|T(x)| \in \{2, 3\}$.

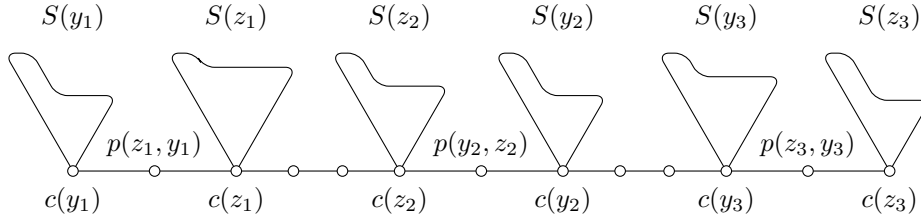
Let us denote by y_1, \dots, y_t and z_1, \dots, z_t the elements of Y and Z that are contained in the triples of $T(x)$, where $t = |T(x)|$. The tree $F(x)$ of the forest F^X is obtained from the copies of the $2t$ stars $S(y_1), \dots, S(y_t), S(z_1), \dots, S(z_t)$ where their centers are connected by a path in which each edge is alternatively replaced by a path of length two or three. Furthermore, along the path, the $2t$ star centers are encountered in the specific order

⁵ Clearly, any element has to appear at least once in T , and if it appears only once, then we have to take it in any solution, and so reducing the instance.



■ **Figure 9** The tree $F(w)$ associated with element $w \in Y \cup Z = \{1, \dots, 2n\}$ constructed from $S(w) \in \mathcal{S}$ and a claw $K_{1,3}$ (in red).

$S(y_1) - \circ - S(z_1) - \circ - \circ - S(z_2) - \circ - S(y_2)$ if $t = 2$ (dashes mean edges), and in the order $S(y_1) - \circ - S(z_1) - \circ - \circ - S(z_2) - \circ - S(y_2) - \circ - \circ - S(y_3) - \circ - S(z_3)$ if $t = 3$. See Figure 10 with the notation therein.



■ **Figure 10** The tree $F(x)$ associated with the element $x \in X$. In this example, $T(x) = \{(x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3)\}$, and $F(x)$ is composed of six distinct stars of \mathcal{S} . Vertex $c(w)$ is the center of the copy of $S(w)$ in $F(x)$, and $p(w, w')$ with $w < w'$ is the unique vertex of the path between the centers $c(w)$ and $c(w')$, if $(x, w, w') \in T(x)$.

Clearly, $F(x)$ is a tree of pathwidth two. Since each star $S(w)$ has $3n + 1$ vertices (cf. Claim 12), the number of vertices of $F(x)$ is $|V(F(x))| = 2t \cdot (3n + 1) + 3t - 2 = 6tn + 5t - 2 \in \{12n + 8, 18n + 13\}$ depending on t . By construction, we have $|V(F^X)| = \sum_{x \in X} |V(F(x))|$.

An important property of $F(x)$ is that if $(x, y, z) \in T$, then a copy of the stars $S(y)$ and $S(z)$ appears as an isometric subgraph of $F(x)$. However, neither $F(y)$ nor $F(z)$ can appear in $F(x)$ as an isometric subgraph because of their claws. Another important property is that if $(x, y, z) \notin T$, then the stars $S(y)$ and $S(z)$ cannot appear consecutively in the main path of $F(x)$ (if they appear). This is due to the specific order of the stars along the path. In that case, the distance between their centers is at least 5.

Let \mathcal{U} be any minimum isometric-universal graph for $\mathcal{F} = \{F^X, F^Y, F^Z\}$. We note that \mathcal{F} has polynomial size w.r.t. the instance (X, Y, Z, T) , since each of the forests F^X, F^Y, F^Z is composed of n trees with $O(n)$ vertices each.

Intuition behind the proof. Each component of \mathcal{U} must contain the trees $F(x)$, $F(y)$ and $F(z)$ for some elements x, y, z . In a favorable case, this component is composed of $F(x)$ plus only one single vertex. Indeed, if y, z are both contained in $T(x)$, then $S(y)$ and $S(z)$ each appears in $F(x)$. It follows that $F(y)$ and $F(z)$ will almost fit in $F(x)$ but for one vertex of their claw. The only way to pack in $F(x)$ using one extra vertex is to share this vertex with their claw by being consecutive on the main path of $F(x)$. This favorable situation will occur

simultaneously on each $F(x)$ if (X, Y, Z, T) has a solution, i.e., if T contains a matching for $X \times Y \times Z$. Unfortunately, if y or z are not in $T(x)$, and still want to pack in $F(x)$, then the resulting component would contain more than one vertex more as in $F(x)$. This situation will necessarily occur if (X, Y, Z, T) has no solution. Overall, the number of vertices in \mathcal{U} will allow us to determine whether a solution exists or not for (X, Y, Z, T) .

More formally, our goal is to prove the following claim:

▷ **Claim 13.** $|V(\mathcal{U})| \leq |V(F^X)| + n$ if and only if (X, Y, Z, T) has a solution.

The proof of this claim is available in the full version [3]. ◀

4 Conclusion and Open Question

We have introduced the notion of k -isometric-universal graph, which captures several well-studied variants of universal graph, by taking different values for k , with $k = 0$ (classical universal graphs), $k = 1$ (induced-universal graphs), or $k = \infty$ (isometric-universal graphs). We have also considered limited families of graphs (by taking two graphs, two trees or forests, all n -vertex graphs, ...).

We have mainly proved that, for $k = \infty$, the problem of minimizing the number of vertices of a minimum k -isometric-universal graph is polynomial for two forests and NP-hard for three forests. The variants for $k \in \{0, 1\}$ are known to be NP-hard.

Along the way, we have proved that any minimum and minimal isometric-universal graph for two trees has to be a tree (Theorem 2). We propose two open questions, one about the structure of minimum isometric-universal graphs, and one about the complexity for two graphs. More precisely,

|| *What is the arboricity of a minimum and minimal isometric-universal graph of $t > 2$ trees? Is it $t - 1$?*

Theorem 2 shows that this is indeed $t - 1$ for $t = 2$. Note that Theorem 8 shows that for $t = 3$ trees, arboricity 1 is not possible.

|| *What is the complexity of finding a minimum isometric-universal graph for two outerplanar graphs? Or two graphs of bounded treewidth?*

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