


Freeze-Tag in L_1 has Wake-up Time Five

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Abstract

The FREEZE-TAG PROBLEM, introduced in Arkin et al. (SODA'02) consists of waking up a swarm of n robots, starting from a single active robot. In the basic geometric version, every robot is given coordinates in the plane. As soon as a robot is awakened, it can move towards inactive robots to wake them up. The goal is to minimize the wake-up time of the last robot, the *makespan*.

Despite significant progress on the computational complexity of this problem and on approximation algorithms, the characterization of exact bounds on the makespan remains one of the main open questions. In this paper, we settle this question for the ℓ_1 -norm, showing that a makespan of at most $5r$ can always be achieved, where r is the maximum distance between the initial active robot and any sleeping robot. Moreover, a schedule achieving a makespan of at most $5r$ can be computed in optimal time $O(n)$. Both bounds, the time and the makespan are optimal. This implies a new upper bound of $5\sqrt{2}r \approx 7.07r$ on the makespan in the ℓ_2 -norm, improving the best known bound so far $(5 + 2\sqrt{2} + \sqrt{5})r \approx 10.06r$.

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1 Introduction

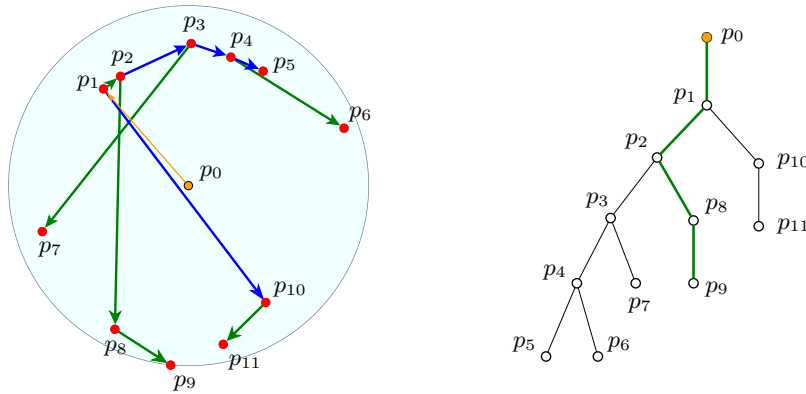
The FREEZE-TAG PROBLEM (FTP) is an optimization problem that consists of activating as fast as possible a swarm of robots represented by points in some metric space (in general, not necessarily euclidean). Active (or awake) robots can move towards any point of the space at a constant speed, whereas inactive robots are asleep (or frozen) and can be activated only by a robot moving to their positions. Initially, there are n sleeping robots and one awake robot. The goal is to determine a schedule whose makespan is minimized; that is, the time until all the robots have been activated is minimized. FTP has application not only in *robotics*, e.g. with group formation, searching, and recruitment, but also in *network design*, e.g. with broadcast and IP multicast problems. See [ABG⁺03, ABF⁺06, KLS05] and references therein.

FTP is NP-Hard in high dimension metrics like centroid metrics [ABF⁺06] (based on weighted star n -vertex graphs) or unweighted graph metrics with a robot per node [ABG⁺03]. Many subsequent works have extended this hardness result to constant dimensional metric spaces, including the Euclidian ones. A serie of papers [AAJ17, Joh17, PdOS23] proves that FTP is actually NP-Hard in (\mathbb{R}^3, ℓ_p) , for every $p \geq 1$, i.e., in 3D with any ℓ_p -norm. For 2D spaces, this remains NP-Hard for (\mathbb{R}^2, ℓ_2) , leaving open the question for other norms [AAJ17].

It is believed, see [ABF⁺06, Conjecture 28], that FTP remains NP-Hard for (\mathbb{R}^2, ℓ_1) .

Several approximation algorithms and heuristics have been designed. In their seminal work, [ABF⁺06] developed a 14-approximation for centroid metrics, and a PTAS for (\mathbb{R}^d, ℓ_p) . [ABG⁺03] presented a $O(1)$ -approximation for unweighted graph metric with one robot per node, and a greedy strategy analyzed in [SABM04] gives a $O(\log^{1-1/d} n)$ -approximation in (\mathbb{R}^d, ℓ_p) . For general metrics, the best approximation ratio is $O(\sqrt{\log n})$ [KLS05]. For heuristics, several experiments results can be found in [Buc04, BHHK07, Kes16]. See [AAS10, MB14, HNP06, BW20] for generalizations and variants of the problem, including the important online version.

As observed by [ABF⁺02], the FTP can be rephrased as finding a rooted spanning tree on a set of points with minimum weighted-depth, where the root node (corresponding to the awake robot) has one child and all the others nodes (corresponding to the n sleeping robots) have at most two children (see Figure 1). Each edge has a *length*, a non-negative real, representing the distance in the metric space between its endpoints. Such a tree is called a *wake-up tree*, and its weighted-depth is called *makespan*.



■ **Figure 1** Example of a (here, euclidean) instance of FTP (on the left). The robot at p_0 must wake up $n = 11$ sleeping robots at p_1, \dots, p_n . In this example, positions are normalized in the unit ℓ_2 -disk, p_0 being at the center. The optimal solution, depicted by arrows, can be represented as a binary weighted tree (right). The makespan is the length of the longest (weighted) branch in that tree, here 2.594, corresponding to the path $(p_0, p_1, p_2, p_8, p_9)$. Observe that, even if the sleeping robots are in a convex configuration, the optimal solution may have multiple edge crossings.

Clearly, FTP is related to the TRAVELING SALESPERSON PROBLEM, in its *metric* version (hereafter, simply TSP). Indeed, the Path-TSP, a generalization of TSP in which one ask for finding a minimum path length spanning a point set from given start and end points, provides a valid wake-up tree and thus a solution for FTP. The link with TSP is reinforced by the recent approximated reduction of Path-TSP to classical TSP [TVZ20]: if there is an α -approximation for TSP, then, for every $\varepsilon > 0$, there is a $(\alpha + \varepsilon)$ -approximation for Path-TSP. Recently [KKOG21] showed that there exists $\alpha < 3/2 - 10^{-36}$ for TSP, a qualitative breakthrough since the 1976 Christofides-Serdyukov algorithm.

This being said, there are significant differences between TSP and FTP, the latter being considered as a cooperative TSP version where awakened robots can help in visiting unvisited cities. First, from an algorithmic point of view, the best lower bound on the approximation factor are $5/3 - \varepsilon$ for FTP [ABF⁺06], and only $123/122 - \varepsilon$ for TSP [Kar15] (assuming $P \neq NP$). On the other side, the time complexity for PTAS in (\mathbb{R}^d, ℓ_p) is $n(\log n)^{O(d/\varepsilon)}$ for TSP [Aro98, RS98, Mit99] vs. $O(n \log n) + 2^{(d/\varepsilon)^{O(d)}}$ for FTP [ABF⁺06], subject to $\varepsilon \leq \varepsilon_d$,

where ε_d depends on the number of dimensions d . Second, and perhaps more fundamentally, it is well known that, even in the unit ball in (\mathbb{R}^d, ℓ_p) , the shortest spanning path (or tour) has unbounded length in the worst-case (it depends on n), whereas the makespan for FTP is bounded by an absolute constant (that does not depend on n). For TSP, the worst-case length is $\Theta_d(n^{1-1/d})$ [Few55], whereas for FTP the worst-case optimal makespan is no more than some constant ρ_d , so independent of n [ABF⁺06].

The constant ρ_d plays an important role for PTAS and approximation algorithms. For instance, it drives the condition “ $\varepsilon \leq \varepsilon_d$ ” in the grid refinement approach of [ABF⁺06], where local solutions in radius- $(1/\varepsilon)$ balls have to be constructed. For (\mathbb{R}^2, ℓ_2) , the constant ρ_2 coming from the approach of [ABF⁺06] has been proved to be at most 57 by [YBMK15]. The latter authors have also constructed in time $O(n)$ a wake-up tree of makespan at most $5 + 2\sqrt{2} + \sqrt{5} \approx 10.06$, which is the best known upper bound for ρ_2 .

Our contributions

In this paper, we concentrate our attention to the plane \mathbb{R}^2 . Given a norm η , the *unit disk w.r.t. η* , or the unit η -disk for short, is the normed linear subspace of (\mathbb{R}^2, η) induced by all the points at distance at most one from the origin, where distances are measured according to η , the distance between u and v being $\eta(v - u)$. The unit η -disk can be an arbitrarily convex body that is symmetric about the origin. Note that the unit ℓ_2 -disk¹ is an usual disk whereas the unit ℓ_1 -disk is a rotated 45 degrees square.

Our main contribution is the following.

► **Theorem 1.** *A robot at the origin can wake up any set of n sleeping robots in the unit ℓ_1 -disk with a makespan of at most 5. The wake-up tree can be constructed in $O(n)$ time.*

Obviously, if the awake robot is at distance at most r from all the sleeping robots, then by scaling the unit disk with their positions, and by using Theorem 1, one can construct a wake-up tree of makespan of at most $5r$. By a loose argument, this yields a 5-approximation $O(n)$ time algorithm for (\mathbb{R}^2, ℓ_1) , since r is a trivial lower bound on the makespan. As we will see, a similar statement holds for $(\mathbb{R}^2, \ell_\infty)$.

Both bounds in Theorem 1 are optimal: the makespan of 5, and obviously the linear time construction of the wake-up tree. The upper bound of 5 is reached with $n = 4$ sleeping robots at positions $(\pm 1, 0)$ and $(0, \pm 1)$. Indeed, any wake-up tree spanning more than four points must have (unweighted) height at least 3. Then, the first hop has length 1 and the next two hops have length 2, which overall gives a makespan of at least 5 for any wake-up tree. Actually, we will see in Theorem 3 a generalization of this argument for any norm η , leading to an intriguing open question of matching this lower bound for other norms (see Conjecture 6).

By a simple argument, Theorem 1 immediately improves the best known upper bound for norm ℓ_2 . Indeed (see also Corollary 5), by scaling the ℓ_2 -disk, we can use the construction of Theorem 1 to obtain a makespan of $5\sqrt{2} \approx 7.07$ for the unit ℓ_2 -disk, improving upon the previous 10.06 upper bound of [YBMK15].

Our second result concerns algorithmic aspects of the FTP. Theorem 2 states that there is always a linear time algorithm that can match the best upper bound for wakening a unit disk.

¹ For convenience, and to avoid extra notation, we use the same “unit η -disk” terminology to denote the normed subspace and, like here, its support, that is the set of all points of norm at most 1 (the disk).

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The result is general enough to hold in any normed linear space (\mathbb{R}^2, η) , akka Minkowski plane.

To make the statement of Theorem 2 precise, let us define $\gamma_n(\eta)$ as the worst-case optimal makespan of a wake-up tree for any set of n sleeping robots in the unit η -disk and rooted at the origin. In other words, $\gamma_n(\eta)$ is the best possible upper bound of the makespan to wake-up n sleeping robots from an awake robot placed at the origin, in the unit η -disk. Finally, let us introduce the *wake-up ratio* w.r.t. the η -norm defined by

$$\gamma(\eta) = \max_{n \in \mathbb{N}} \gamma_n(\eta) .$$

Note that the constant ρ_2 introduced above is nothing else than $\gamma(\ell_2)$, the ℓ_2 wake-up ratio.

► **Theorem 2.** *Let η be any norm and let $\tau > 3$ be any real such that $\tau \geq \gamma(\eta)$. Knowing τ , one can construct in time $O(n)$ a wake-up tree of makespan at most $\gamma(\eta)$ for any set of n points in the unit η -disk and rooted at the origin.*

So, plugging $\eta = \ell_1$ and $\tau = 5$ in Theorem 2, it is sufficient to prove that $\gamma(\ell_1) \leq 5$ to automatically obtain a linear time construction of a wake-up tree of makespan at most 5 as claimed in Theorem 1. In other words, given Theorem 2, our main Theorem 1 could simply be restated as: $\gamma(\ell_1) \leq 5$. Furthermore, as already explained, the bound of 5 is attained for $n = 4$ sleeping robots, so $\gamma(\ell_1) \geq \gamma_4(\ell_1) = 5$, and the wake-up ratio in ℓ_1 -norm is thus 5.

To prove Theorem 1 and Theorem 2, we need several intermediate results, which we believe are of independent interest. For instance, we show how to efficiently wake up robots contained in a unit cone of given arc-length. This implies, for instance, that $\gamma(\eta) < 3 + 4\varphi = 5 + 2\sqrt{5} \approx 9.47$, where φ is the golden ratio.

Some of our arguments rely on an intermediate result that we prove for all possible norms. Given a norm η , let us define $\Lambda(\eta)$ as half the perimeter of the largest inscribed parallelogram in the unit η -disk (in ℓ_1 -norm, this perimeter is the disk itself). This is a classical parameter for normed spaces. It can be formally defined by (see [Sch76, Gao01]),

$$\Lambda(\eta) = \sup_{\substack{u, v \in \mathbb{R}^2 \\ \eta(u), \eta(v) \leq 1}} \{ \eta(u + v) + \eta(u - v) \} .$$

It is easy to check that $\Lambda(\eta) \in [2, 4]$. For general norms, the quantity $\Lambda(\eta)$ is difficult to calculate. However, it is known (see [Gao01, Proposition 1] for instance), that, for every $p \in [1, \infty]$, $\Lambda(\ell_p) = 2^{1+\max(1/p, 1-1/p)}$. In particular, $\Lambda(\ell_1) = \Lambda(\ell_\infty) = 4$ and $\Lambda(\ell_2) = 2\sqrt{2}$.

The wake-up ratios for $n \in \{0, 1, 2, 3\}$ are easy to calculate. We have $\gamma_0(\eta) = 0$, $\gamma_1(\eta) = 1$, $\gamma_2(\eta) = \gamma_3(\eta) = 3$, and also $\gamma_n(\eta) \geq 3$ for all² $n \geq 3$. Our next result gives the exact value for $\gamma_4(\eta)$.

► **Theorem 3.** *For any norm η , $\gamma_4(\eta) = 1 + \Lambda(\eta)$.*

This implies a general lower bound of $\gamma(\eta) \geq 1 + \Lambda(\eta)$. Note that since $\Lambda(\eta) \geq 2$, Theorem 2 simplifies and rewrites in:

► **Corollary 4.** *For any norm η with $\Lambda(\eta) \neq 2$, one can construct in time $O(n)$ a wake-up tree of makespan at most $\gamma(\eta)$ for any set of n points in the unit η -disk and rooted at the origin.*

² For $n \geq 2$, it is enough to place one sleeping robot at $(1, 0)$ and the $n - 1$ others at $(-1, 0)$.

Now, combining Theorem 1, Theorem 3 and standard inclusion arguments of unit ℓ_p -disk, we get the following bounds for the ℓ_p wake-up ratio:

► **Corollary 5.** *For every $p \in [1, \infty]$, $1 + 2^{1+\max(1/p, 1-1/p)} \leq \gamma(\ell_p) \leq 5 \cdot 2^{\min(1/p, 1-1/p)}$.*

In the light of the lower bound $\gamma(\eta) \geq 1 + \Lambda(\eta)$ implied by Theorem 3, we propose the following natural conjecture.

► **Conjecture 6.** *For any norm η , $\gamma(\eta) = 1 + \Lambda(\eta)$.*

According to Theorem 3, which states that the bound $1 + \Lambda(\eta)$ is reached by $n = 4$ robots, Conjecture 6 can be captured in the aphorism:

It's always quicker to wake up n robots than four.

Theorem 1 and Corollary 5 prove the conjecture for $\eta \in \{\ell_1, \ell_\infty\}$. For $\eta = \ell_2$, if true, Conjecture 6 combined with Theorem 2 imply that in time $O(n)$ one can construct a wake-up tree of makespan $1 + 2\sqrt{2} \approx 3.82$. Proposition 15 implies also that Conjecture 6 is true for ℓ_2 whenever $n \geq 528$.

2 The wake-up ratio is at most 5 in L_1

Our main result (Theorem 1) is to prove that the wake-up ratio for the ℓ_1 -norm is at most 5. The proof is constructive and provides a polynomial time algorithm. The complexity is subsequently improved to a $O(n)$ in Section 3.

At a high level, the strategy consists of recruiting first a team of robots in a dense subregion, then these robots can wake up the other regions in parallel. The difficult part is to select these regions appropriately, depending on the number of robots and their distribution, and to prove that the bound holds in all the cases.

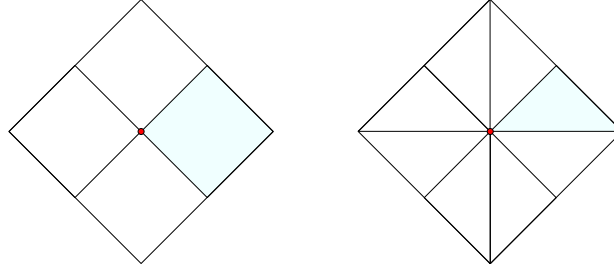
To make this more precise, let us partition the unit ℓ_1 -disk into *squares* and *triangles* as follows. A *square* (of diameter 1) is a square region whose diagonals and sides are both of length 1 (ℓ_1 norm) and the diagonals are parallel to the x -axis and y -axis (see Figure 2). Similarly, a *triangle* (of diameter 1) is an isosceles right triangle whose hypotenuse and sides have length 1 and the hypotenuse is parallel to the x -axis or y -axis. Thus, each square region represents a fourth of the unit ℓ_1 -disk, possibly subdivided further into two equal triangles. Strictly speaking, the diameter may be smaller than 1, as our algorithm occasionally subdivides some regions further, but the arguments are then normalized to 1 systematically. Finally, note that both squares and triangles can be seen as *cones* (of different angles) in the ℓ_1 -norm.

The algorithm relies crucially on three lemmas about these regions, namely:

► **Lemma 7.** *A robot located at a corner of a square can wake up any number $n \leq 5$ of robots in the square in two time units.*

► **Lemma 8.** *A robot located at a corner of a square can wake up 6 robots in the square and return to the origin with these robots in three time units.*

► **Lemma 9.** *A robot located at any of the three corners of a triangle T , or two robots located at a same point on a side of T (not the hypotenuse) can wake up all the robots in T in two time units.*



■ **Figure 2** The unit ℓ_1 -disk, divided into squares and triangles of diameter 1.

A significant part of the paper is devoted to proving these lemmas. In particular, Lemma 9 is proved by an induction involving 14 subcases. Equipped with these lemmas, the algorithm can be described in a compact way as follows.

The algorithm. The strategy is split into four scenarios as follows, depending on the number n_0 of robots in the densest square.

- $n_0 = 1$. In this case, there are at most four robots to be awakened. The initiator wakes up one of them in one time unit. We now have two awake robots. Each of them independently wakes up another sleeping robot, in at most two time units (largest possible distance within the unit ℓ_1 -disk). Then, any of the awake robot wakes up the last robot in at most two time units, which gives a total makespan of at most $1 + 2 + 2 = 5$.
- $2 \leq n_0 \leq 5$. We recruit n_0 robots from the densest square S in two time units (Lemma 7), then come back to the origin (by time $2 + 1 = 3$) with $n_0 + 1 \geq 3$ awake robots. Since S is the densest square, then three of the awake robots at the origin can each wake up one of the remaining squares (Lemma 7) in two time units, which gives a total of at most $3 + 2 = 5$.
- $6 \leq n_0 \leq 10$. We recruit 6 robots (chosen arbitrarily) in the densest square S and move them to the origin in 3 time units (Lemma 8). Together with the initiator, this makes 7 robots. One of them wakes up the remaining robots in S , which are at most 4, in two time units (Lemma 7). The 6 others split into three teams of two robots, one team for each remaining square, and each robot wakes up half of the sleeping robots in its assigned square, again in two time units (Lemma 7), which gives a total of at most $3 + 2 = 5$.
- $n_0 \geq 11$. The densest square S must contain a triangle T with at least $\lceil n_0/2 \rceil \geq 6$ sleeping robots. We wake up all the robots of T in 2 time units (Lemma 9) and move them to the origin. This makes at least 7 robots. Each of them wakes up a remaining triangle in 2 time units (Lemma 9 again), which gives a total of at most $2 + 1 + 2 = 5$.

And this completes the proof of Theorem 1. ◀

2.1 Preamble

The positions of robots are given as a multiset of points $\{p_0, p_1, \dots, p_n\}$ taken in the unit ℓ_1 -disk, where p_0 is the position of the awake robot, and p_1, \dots, p_n the positions of the n sleeping robots. We use the notation $|p_i p_j|$ to denote the ℓ_1 -distance between p_i and p_j , i.e., $|p_i p_j| = \ell_1(p_j - p_i) = |x(p_j) - x(p_i)| + |y(p_j) - y(p_i)|$.

We define two orderings for points $p, q \in \mathbb{R}^2$. Namely, $p \leq_x q$ if the x -coordinate of p is no more than the one of q . We also say the p is on *the right* of q (or that q is on *the left* of p). Similarly, $p \leq_y q$ if the y -coordinate of p is no more than the one of q , and we say that p is *below* q (or that q is *above* p).

Monotonic paths

A path (p_0, p_1, \dots, p_t) , $t \geq 1$ is *monotonic* if it is compatible with both \leq_x and \leq_y . More precisely, for each $i \in \{1, \dots, t\}$, we must have $(p_{i-1} \leq_x p_i \Leftrightarrow p_0 \leq_x p_1)$ and $(p_{i-1} \leq_y p_i \Leftrightarrow p_0 \leq_y p_1)$. In other words, monotonic paths use points that are always going in the same direction w.r.t. the four quadrants: North-East (NE), North-West (NW), South-West (SW), South-East (SE).

A fundamental property of the ℓ_1 -norm is that all monotonic paths are shortest paths. More formally, if (p_0, p_1, \dots, p_t) is a monotonic path, then

$$\sum_{1 \leq i \leq t} |p_{i-1} p_i| = |p_0 p_t|.$$

A path is *k-monotonic* if it can be subdivided into k monotonic subpaths. We thus have:

► **Lemma 10.** *In a region of diameter D , the length of a k -monotonic path is at most kD .*

Our algorithm exploits monotonic paths on several occasions, in order to wake up intermediate robots at no additional costs.

► **Lemma 11.** *Any set of 5 points (or more) contains a monotonic path of length 3.*

Proof. By the Erdős-Szekeres theorem, given α and β , any sequence of distinct numbers of length at least $(\alpha - 1)(\beta - 1) + 1$ contains a monotonically increasing subsequence of length α or a monotonically decreasing subsequence of length β . In two dimensions, one can first order the points according to \leq_x , then consider the y -coordinates as the sequence of interest. The result follows by taking $\alpha = \beta = 3$. ◀

In the following subsections, we show how to wake up between 1 and 6 robots in a square S of diameter 1 starting at one of its corner. For technical reasons, we must distinguish the case $n \leq 5$ with makespan 2 (Lemma 7) and the case $n \leq 6$ with makespan 3 including the return to origin (Lemma 8). Then, we give part of the proof of Lemma 9, the remaining part being deferred to Appendix A.

2.2 Proof of Lemma 7

► **Lemma 7.** *A robot located at a corner of a square can wake up any number $n \leq 5$ of robots in the square in two time units.*

Proof. Without loss of generality, suppose that the awake robot is located at a point p_0 at the left corner of the square. Thus, all the sleeping robots are on its right. We deal with a few cases separately:

- $n \leq 3$: Any wake-up tree of depth 2 works, since the diameter of S is 1.

- $n = 4$: By Lemma 11, S contains three robots whose positions define a monotonic path (p_i, p_j, p_k) of length three (including possibly p_0). If we omit p_j , then by the previous case, any wake-up tree has makespan 2. Pick a tree where one of the branch goes from p_i to p_k . Then, we can insert p_j on the way from p_i to p_k without impacting the makespan.
- $n = 5$: Either p_0 belongs to a monotonic path, or it does not. If it does, the initial robot wakes up the two corresponding robots in a single time unit, which gives us three awake robots, each of which wakes up one of the remaining robots in one time unit. If it does not, then among the sleeping robots, there is a unique robot p^+ whose y -coordinate is maximum, and a unique robot p^- whose y -coordinate is minimum. Call p_1, p_2 , and p_3 the positions of the three remaining robots. Wlog, p_1 is leftmost among these points, and $p_2 \leq_y p_3$. Now, if $p_2 \leq_y p_1 \leq_y p_3$, then (p_0, p_1, p_2) or (p_0, p_1, p_3) is a monotonic path from p_0 . Thus, p_1 must also be topmost or bottommost. Wlog again, suppose that it is topmost and recall that it is also leftmost. If $p_1 \leq_y p_0$, then again (p_0, p_1, p_2) is a monotonic path from p_0 , so $p_1 \geq_y p_0$, and since (p_0, p_1, p^+) cannot be a monotonic path, we also have that $p_1 \geq_x p^+$. This implies that both (p^+, p_1, p_2) and (p^+, p_1, p_3) are monotonic paths. Based on these facts, the wake-up tree consists of first waking up p^+ , resulting in two robots. One of them wakes up p^- , the other wakes up p_1 . Then, one robot at p_1 wakes up p_2 and the other wakes up p_3 . By monotonicity, the sleeping robots at p_2 and p_3 are both woken up within one time unit from p^+ (through p_1). ◀

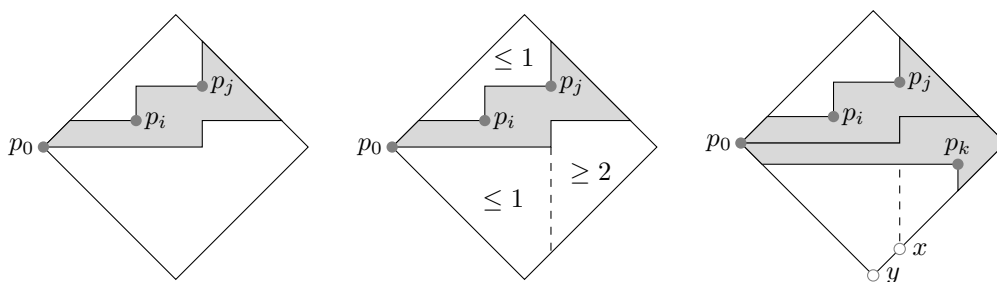
Note that Lemma 7 is best possible in the sense that if S contains exactly 6 sleeping robots, then a makespan of 2 may not be achievable, which motivates the distinction between Lemma 7 and Lemma 8. The reader interested in this fact can have a look at Proposition 17 and Figure 13 in Appendix F (which is independent from our other results).

2.3 Proof of Lemma 8

► **Lemma 8.** *A robot located at a corner of a square can wake up 6 robots in the square and return to the origin with these robots in three time units.*

Proof. Again, suppose that the awake robot is located at the left corner (point p_0). We distinguish three cases:

- **Case 1.** There exists a monotonic path (p_0, p_i, p_j, p_k) . By Lemma 10, we can have four awake robots located at p_k in one time unit. Three of them can wake up the remaining three robots (separately) in one time unit, then each robot can move to p_0 in one time unit.
- **Case 2.** Case 1 does not hold but one or several monotonic paths (p_0, p_i, p_j) exist. Among these, choose one that maximizes the x -coordinate of p_j , and among these (if several), choose one that minimizes the distance between the y -coordinate of p_i and the y -coordinate of p_0 . Wlog, assume that this path is monotonic in the NW direction. Since we are not in Case 1, no other robot may have a position p_k that would cause (p_0, p_k, p_i) or (p_i, p_k, p_j) or (p_i, p_j, p_k) to be monotonic, nor (p_0, p_i, p_k) to be monotonic with $p_k \geq_x p_j$ because p_j is the rightmost such node. The forbidden regions are depicted in gray in the figure below (left). Note that this separates the authorized regions into an upper and a lower region (in white).



Among the remaining points (whatever placed), either two points p and p' exist such that $P_1 = (p_i, p, p', p_0)$ or $P_2 = (p_j, p, p', p_0)$ is a 2-monotonic path, or no such pairs exist. If it exists, then the wake-up tree is as follows. The robot at p_0 wakes up p_i and p_j in one time unit. The robot at p_i stays at p_i , so we have one robot at p_i and two at p_j . Then, depending on whether P_1 or P_2 exists (if both exist, pick any), the corresponding robot wakes up p and p' and return with them at the origin. The other two wake up the last two robots (independently), and return with them at the origin. Overall, each robot has moved along a path that is at most 3-monotonic.

If neither P_1 nor P_2 exist, the strategy is different. First, observe that having two or more robots in the upper region would create a 2-monotonic path from either p_i or p_j to p_0 through these robots, thus the upper region contain at most one robot. By the same argument (from p_j alone), the part of the lower region at the left of p_j also contains at most one robot, so the situation is as depicted on the figure (middle). In particular, the lower right region is not empty. Let p_k be the position of the rightmost robot in this region. Only one robot has such a x -coordinate, as otherwise, p_j would have had a 2-monotonic path through them towards p_0 . For the same reason, none of the remaining robots are above p_k , and since p_k is rightmost, no robots are on its right either, see the figure (right) for the remaining possible zones. Apart from p_k , the lower right region has between 1 and 3 robots; however, if they are at least 2, then they must be aligned along a SW monotonic path from p_k , as otherwise (again), p_j would have a 2-monotonic path through them. Thus, a single monotonic path from p_k can wake up all the robots in the lower right region. Finally, if there is a robot in the upper region, then it is possible to find a 2-monotonic path from p_k to p_0 going through p_j and this robot. Now that all these facts are stated, the wake-up tree is as follows. Here, the robot at p_0 wakes up p_k first. Then, using a 2-monotonic path, one of the robot wakes up p_j and the potential robot in the upper region, and return at p_0 . The second robot at p_j wakes up p_i on its way to p_0 . Meanwhile, the second robot at p_k wakes up all the robots in the lower right region using a monotonic path that finishes either at x or y (see the figure), depending on where the potential robot in the lower left region lies. This robot is woken up and all the robots return to p_0 . All these movements are made along paths that are at most 3-monotonic.

- **Case 3.** No monotonic path of the form (p_0, p_i, p_j) exists. In this case, all the points below (resp. above) the y -coordinate of p_0 must form a 1-monotonic path in the NE/SW direction (resp. NW/SE direction). In this case, we wake up the rightmost robot first. Then, one of the two robots wakes up the upper robots (if any) and the other wakes up the lower robots (if any). Finally, they all move to p_0 . All these movements are made along paths that are at most 3-monotonic.

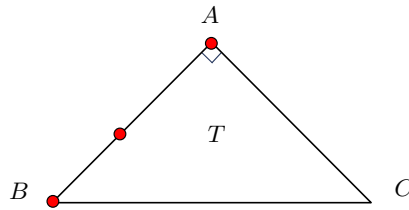


2.4 Proof of Lemma 9

Lemma 9 establishes that an arbitrary number of sleeping robots in a triangle T can be woken up within two time units. The approach is inductive, namely, waking up a triangle often reduces to waking up smaller nested triangles (containing strictly less robots), which explains why the formulation of the lemma addresses several starting configurations.

► **Lemma 9.** *A robot located at any of the three corners of a triangle T , or two robots located at a same point on a side of T (not the hypotenuse) can wake up all the robots in T in two time units.*

Without loss of generality, we assume that the triangle T is oriented as in Figure 3, with vertices ABC and hypotenuse $[BC]$. The goal is to show that all sleeping robots in T can be



■ **Figure 3** The triangle T with vertices $B = (0, 0)$, $C = (1, 0)$ and $A = (1/2, 1/2)$. In red, the possible starting points covered by the lemma.

woken up in two time units, for each of the possible starting configurations. Up to symmetry, these configurations are:

- Case A.** One awake robot is located in A .
- Case B.** One awake robot is located in B .
- Case C.** Two awake robots are located at a same point along segment $[AB]$.

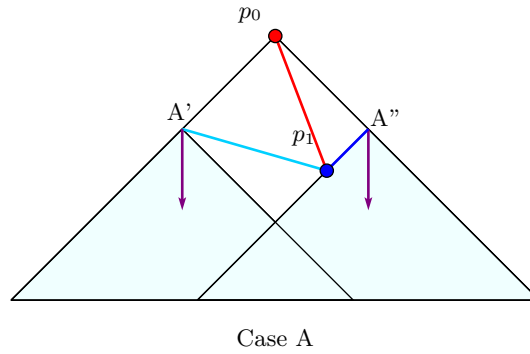
The strategy depend critically on how the robots are distributed within the triangle, which gives rise to a number of subcases (14 overall). We agree that case-based proofs are not always satisfactory. However, our proof is at least fully constructive (i.e., it yields an actual algorithm). Furthermore, it is plausible that obtaining tight bounds for this problem requires an unavoidable low-level scrutiny of the instance. Indeed, many of the cases achieve the bound in a tight way. We now proceed with the main three cases.

2.4.1 Case A

Here, the awake robot is located at the top of the triangle (point A , called p_0). This case does not rely on the same subdivisions as above. It uses a simpler recursion to smaller instances of Case A again, as shown in Figure 4.

Let p_1 be the closest sleeping robot from p_0 . Let $d = |p_0 p_1|$, and let S_A be the smallest square in T that contains both A and p_1 . Due to the ℓ_1 -norm, this square has diameter d . Furthermore, it is empty because p_1 is the closest point to A . Let A', A'' be the points of S_A intersecting $[AB]$ and $[AC]$, respectively, and let T' and T'' be the triangles defined homothetically to T with respect to A' and A'' . These triangles have diameter $1 - d$.

The wake-up strategy is as follows. The robot at p_0 wakes up the robot at p_1 . Then, one goes to A' in order to wake up the robots in T' , the other goes to A'' to wake up the robots

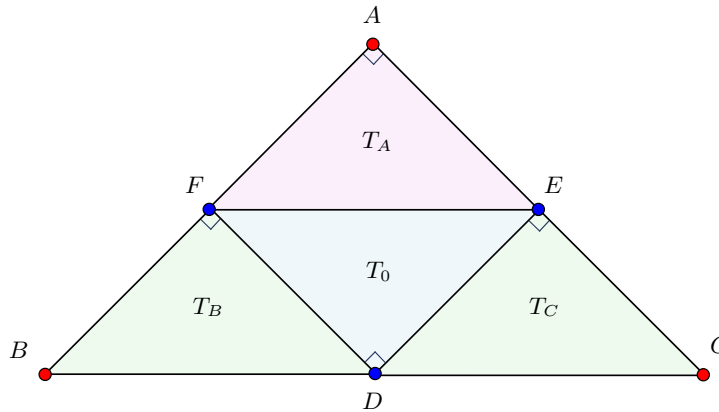


■ **Figure 4** In case A, the robot located at the top awakes the closest sleeping robot, then each of them applies recursively the algorithm in one of the two subtriangles.

in T'' (breaking ties arbitrarily if $T' \cap T''$ is not empty). Since S_A has diameter d , any 2 hops path has length at most $2d$, so the two robots reach A' and A'' before that time. Then, T' and T'' are woken up in parallel (recursion of Case A), in at most $2(1 - d)$ time unit, which gives a total of $2d + 2(1 - d) = 2$ time units.

2.4.2 Cases B

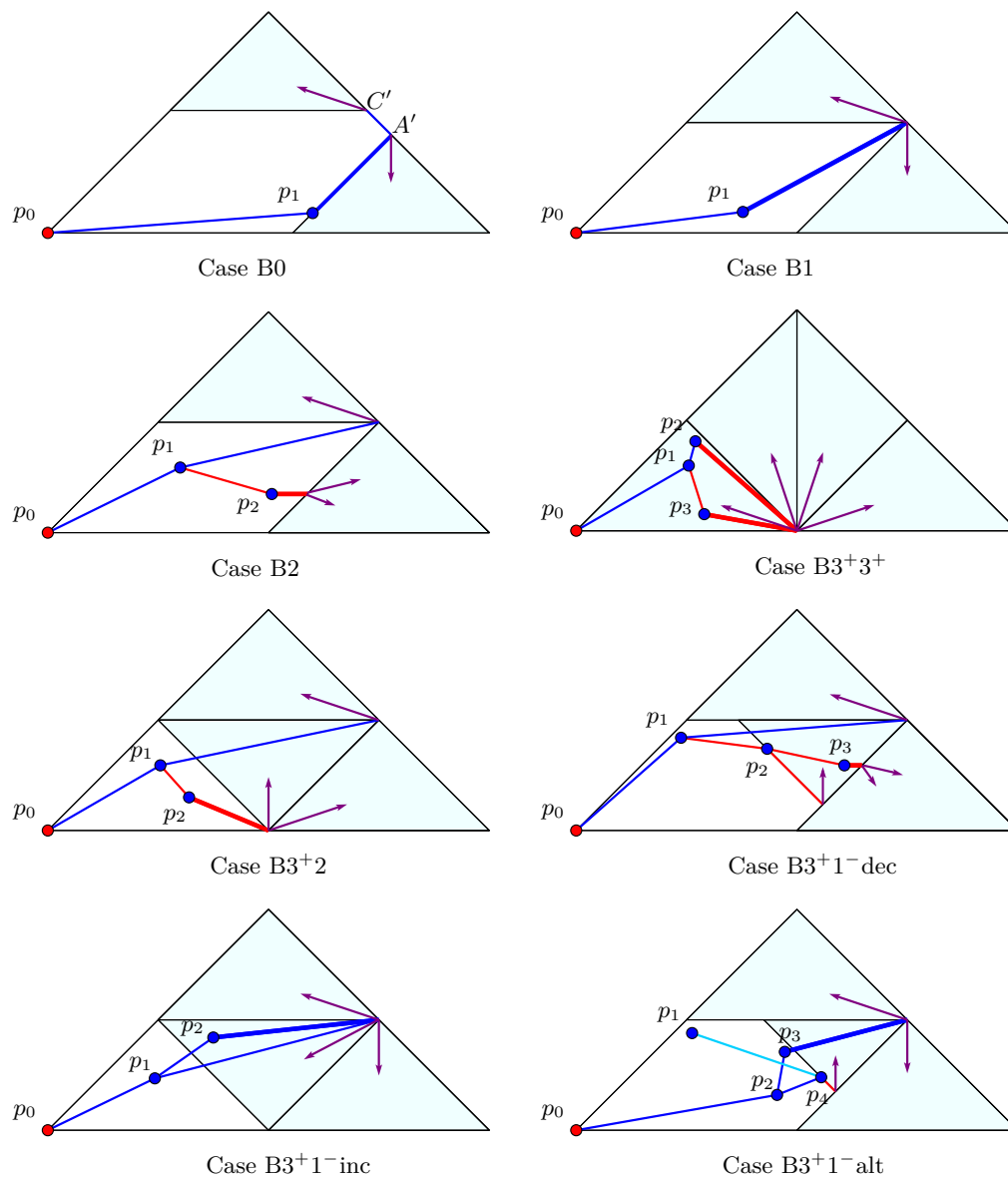
The proof of Case B (and Case C) rely on a regular subdivision of T into four smaller triangles of equal size. Call D, E, F the middle points of segments $[BC], [CA]$ and $[AB]$, respectively, and let T_A, T_B, T_C, T_0 be the triangles AFE, BDF, CED , and DEF (see Figure 5). Each of these triangles has diameter $1/2$. Similarly, let P_A and P_B be the two parallelograms $AEDF$ and $BDEF$. The diameter of P_A is $1/2$, and the one of P_B is 1.



■ **Figure 5** Canonical subdivision of the triangle T , with vertices $B = (0, 0)$, $C = (1, 0)$ and $A = (1/2, 1/2)$.

Recall that in Case B, the awake robot start at point B , also referred to as p_0 . The case analysis depends on the distribution of nodes in the region defined in the above subdivision, in particular the number of robots in P_B and T_B . A graphical summary of the subcases is shown in Figure 6. The reader is encouraged to come back to these pictures regularly.

The first few cases depend on the number of sleeping robots in P_B . Namely, we apply B_0 if it is empty, B_1 if it contains one robot, and B_2 if it contains two robots.



■ **Figure 6** The 8 subcases of Case B (Lemma 9). Regions in blue correspond to region where recursion occurs. Outgoing purple arrows indicates that the region will be woken up from the head location. A thick edge indicates that two awake robots follow the same path.

- **Case B0.** P_B is empty. We increase the size of P_B homothetically, keeping one of its corners at B , until a point p_1 is found (see Figure 6 - B0). The new parallelogram intersects with $[AC]$ in two points C' and A' , where C' is the highest (i.e. closest to A). This forms two smaller triangles which are homothetical to ABC . Because they result from intersecting a parallelogram, these triangles have the same size; namely, they have diameter $d = |AC'| = |A'C|$, which also implies that $|C'A'| = 1 - 2d$.

The wake-up tree is as follows. The initial robot wakes up p_1 . Depending on what side of the parallelogram p_1 lies on, both robots reach C' or A' using a path that is still monotonic from p_0 , so they arrive before one time unit. One of them then reaches the other point (C' or A') in time $1 - 2d$. Finally, each robot wakes up one of the two triangles (separately) in time $2d$, recursing into case A and B (respectively). Overall, the makespan is thus $1 + (1 - 2d) + 2d = 2$.

- **Case B1.** P_B contains one robot. The robot at p_0 wakes up this robot, then both robots move to E before one time unit, since the path (p_0, p_1, E) is monotonic. Finally, one of them wakes up T_A (recursing in Case B) and the other T_C (recursing in Case A). These triangles have half the size of T , thus the makespan is at most $1 + 2(1/2) = 2$.
- **Case B2.** P_B contains two robots at p_1 and p_2 . Wlog, assume $p_1 \leq_x p_2$. If (p_0, p_1, p_2) is monotonic, the strategy is the same as in Case B_1 : the robots reach point E in one time unit, then two of them wake up T_A and T_C independently. Otherwise, there exists a point C^* of $[DE]$ such that (p_1, p_2, C^*) is 1-monotonic. In this case, the initial robot wakes up the robot in p_1 , then moves to E before one time unit and wakes up T_A in $2(1/2) = 1$ time unit. Meanwhile, the robot in p_1 wakes up the robot in p_2 and both move to C^* .

Claim: The 2-monotonic path $P = (p_0, p_1, p_2, C^*)$ has length at most one.

Proof: Let $p_1 = (x, y)$ and $C^* = (x', y')$. By 2-monotonicity, the length of the path is $|Bp_1| + |p_1C^*| = (x + y) + ((x' - x) + (y - y')) = 2y + x' - y'$. In terms of y -coordinate, the height of T is $1/2$, thus the height of P_B is $1/4$, and $y \leq 1/4$. Moreover, because C^* lies on $[DE]$, we have $y' = x' - 1/2$, so $2y + x' - y' \leq 1/2 + x' - (x' - 1/2) = 1$.

We thus have two robots located at C^* before one time unit. These robot can wake up T_C (of diameter $1/2$) in one time unit, by recursing in Case C.

The remaining cases address the configurations where P_B contains at least three robots. Here, we distinguish based on the number of robots in the subtriangle T_B , namely whether T_B contains three or more robots ($B3^+3^+$), two robots ($B3^+2$), or only zero or one robot ($B3^+1^-$).

- **Case $B3^+3^+$.** T_B contains at least three robots. We consider a slightly different subdivision of the part covered by triangles T_A and T_0 , dividing the corresponding area vertically into two equal triangles ADF and ADE . Let p_1, p_2 and p_3 be the first three points with respect to \leq_x . The wake-up tree is as follows. The robot in p_0 wakes up the robot in p_1 . Then, one of the two goes to p_2 and the other goes to p_3 . Then, the four robots gather at D . Observe that all these paths from p_0 to D are 2-monotonic, and T_B has diameter $1/2$, thus the robots arrive at D before one time unit. Finally, each robot separately wakes up one of the triangles (of diameter $1/2$) in one time unit, by recursing in Case B.
- **Case $B3^+2$.** T_B contains two robots. Let p_1 and p_2 be the first two points with respect to \leq_x . The wake-up tree is as follows. The robot at p_0 wakes up the robot at p_1 . One of the robots goes directly to E , the other wakes up p_2 and the two resulting robots move

to D . Observe that the path (B, p_1, E) is 1-monotonic, thus it has length 1. The path (B, p_1, p_2, D) is 2-monotonic within a triangle of diameter $1/2$, so it has length 1 as well. Finally, the robot in E wakes up T_A (recurring in Case B), one of the robots at D wakes up T_0 (recurring in Case A), and the last robot wakes up T_C (Case B again). All these triangles have diameter $1/2$, thus these recursive operations will take at most another time unit.

- **Case $B3^+1^-$.** T_B contains 0 or 1 robot. We have three subcases. For simplicity, we assume that T_B contains exactly one sleeping robot, thus T_0 contains two or more robots. The arguments are identical if T_B is empty and all the robots of P_B are in T_0 .

- **Subcase $B3^+1^-$ -dec.** There exists a monotonic path (p_1, p_2, p_3) in the SE direction that contains the point in T_B . Let T'_0 be the triangle resulting from shrinking T_0 homothetically (keeping it anchored at E) until at least two of these points lie outside or along the side of T'_0 (Figure 6 - $B3^+1^-$ -dec). Call $D' \in [DE]$ the apex of T'_0 . The wake-up tree is as follows. The robot at p_0 wakes up the robot at p_1 . One of them goes to E , the other wakes up p_2 . Then, one of the robots in p_2 wakes up p_3 and the other goes to D' . Finally, the two robots in p_3 move to any point C^*X along $[DE]$ such that $(p_0, p_1, p_2, p_3, C^*)$ is 2-monotonic. From these locations, the algorithm recurses as follows: the robot in E wakes up T_A (Case B); the robot in D' wakes up T'_0 (Case A); and the two robots in C^* wake up T_C (Case C). Since the path (p_0, p_1, E) is 1-monotonic, and T_A has diameter $1/2$, T_A will be woken up within another time unit, for a total of 2 time units. Furthermore, both paths (p_0, p_1, p_2, D') and $(p_0, p_1, p_2, p_3, C^*)$ are 2-monotonic within P_B . By the same argument as the claim in Case B2, both paths have length at most 1. Thus, T_C (of diameter $1/2$) and T'_0 (whose diameter is at most $1/2$) will also be woken up within two time units overall.
- **Subcase $B3^+1^-$ -inc.** There exists a monotonic path $P = (p_0, p_1, p_2)$ in the NE direction. In this case, p_0 wakes up p_1 . One of them goes to E , the other wakes up p_2 and the two resulting robots go to E . From E , the three robots separately wake up T_A (Case B), T_0 (Case B), and T_B (Case A). By monotonicity, all of them arrive at E before one time unit, and the three subtriangles have diameter $1/2$, thus the overall makespan is 2.
- **Subcase $B3^+1^-$ -alt.** If we are neither in subcase $B3^+1^-$ -dec nor $B3^+1^-$ -inc, then the leftmost three points $p_1 \leq_x p_2 \leq_x p_3$ are such that $p_2 \leq_y p_1$, $p_2 \leq_y p_3$, and $p_3 \leq_y p_1$. Thus (p_0, p_2, p_3, E) is 1-monotonic. The wake-up tree is as follows. The robot at p_0 wakes up p_2 . Then, one of the two robots at p_2 wakes up p_3 and the two resulting robots move to E , where they wake up T_A (recurring in Case B) and T_C (Case A). We are left with a robot at p_2 . If P_B contained exactly 3 robots, then this robot wakes up p_1 . Otherwise, let T'_0 be the triangle obtained by shrinking T_0 homothetically (keeping it anchored at E), until a new point p_4 lies on its side, and let D' be the apex of T'_0 . In this case, the path (p_0, p_2, p_4, D') is 2-monotonic within parallelogram P_B , thus it has length at most 1 (again, by the same claim as in Case B2). Thus, the robot at p_2 wakes up p_4 . One of the resulting robot wakes up p_1 , while the other move to D' and wakes up T'_0 by recurring in Case A. Since T'_0 has diameter at most $1/2$, the overall makespan is again at most 2.

2.5 Case C

Due to space limitations, the proof of case C is deferred to Appendix A. This proof is in the same spirit as the proof of Case B and it also relies on the subdivision shown in Figure 5.

3 Linear time algorithm

In Section 2, we proved that the wake-up time of a unit ℓ_1 -disk can always be upper bounded by 5 time units. The proof was constructive, but its time complexity is not linear. In this section, we prove that a linear time algorithm can asymptotically be achieved. More precisely, there exists a threshold n_0 such that if the number of sleeping robots n is larger than n_0 , then a wake-up tree of makespan less than 5 can be computed in linear time in n (Theorem 2). Thus, whenever $n < n_0$, one can use the constructive procedure from Section 2, then for larger values, one can use the linear time algorithm. Since n_0 is a constant, the computation time when $n < n_0$ is bounded by a constant, which implies that this combined strategy, overall, is a linear time algorithm. Due to space limitation, the content of this section is deferred to Appendix B.

4 Conclusion

We have showed that in linear time one can produce a wake-up tree of makespan at most five for robots in L_1 . This wake-up ratio “five” is optimal: no strategy can guarantee less than five times the radius under the ℓ_1 -norm. For ℓ_2 -norm, we have improved the best known bound from 10.06 to 7.07. Some of our results are general enough to apply to every norm. We have also showed how to get in linear time a wake-up tree of makespan no more than the wake-up ratio, for every norm.

Along the way, we have proposed a conjecture saying that, for every norm η , the wake-up ratio is $1 + \Lambda$, where Λ is half the perimeter of the largest inscribed parallelogram of the unit disk in (\mathbb{R}^2, η) . According to our results, the conjecture is equivalent to saying that it is always quicker to wake up n robots than four. We have proved it for ℓ_1 and ℓ_∞ norms.

As a first step towards this conjecture, it would be interesting to determine the status of the ℓ_2 -norm whose wake-up ratio, according to our conjecture, should be $1 + 2\sqrt{2} \approx 3.82$. Among ℓ_p -norms, ℓ_2 is the norm whose gap between our upper and lower bounds on the wake-up ratio is the largest. In spite of our efforts, we were unable to prove that, for instance, the worst-case situation is whenever the points are all on a circle, and/or equally distributed on the circle. One of the difficulty might be that the longest branch, in a optimal (or near optimal) wake-up tree, does not necessarily form a convex set.

We have showed that the wake-up ratio for fixed n asymptotically decreases with n , i.e., $\gamma_n(\ell_2) < \gamma_4(\ell_2)$ for large n (more than 500), but we were unable to show that this inequality occurs for small n , say n about 10. Surprisingly, experiments we have performed (see Table 1 in Appendix C) show that one (at least) of the two following likely statements must be wrong: (1) the wake-up ratio is reached for points that are equally distributed on the unit circle; (2) for every $n \geq 4$, $\gamma_{n+2}(\ell_2) < \gamma_n(\ell_2)$.

It might be difficult to find the exact bound of the wake-up ratio for ℓ_2 , in the light of other constants in Computational Geometry. This is notably the case for the *stretch factor* of the Delaunay triangulations, the maximum ratio between the distance between any two points in the triangulation and their ℓ_2 -distance. Despite a lot of efforts, current lower and

upper bounds for this stretch factor are 1.593 [BDL⁺11] and 1.998 [Xia13]. Gaps have been closed for C -Delaunay triangulations (defined by some empty convex shape C), only for some specific C , namely for $C \in \{\text{triangle, square, hexagon}\}$, see [Che89, BGHP15, DPT21] respectively.

We summarize a list of further works:

- Calculate the wake-up ratio in ℓ_1 or ℓ_2 for a fixed number of $n > 4$ of sleeping robots.
- Prove or disprove that the wake-up ratio of the ℓ_2 -norm is $1 + 2\sqrt{2}$.
- Prove or disprove that the wake-up ratio of the regular-hexagonal-norm³ is 4.
- Prove or disprove Conjecture 6 for ℓ_p -norms.
- Prove or disprove Conjecture 6 for general norms.
- Construct a linear time PTAS.
- Extend the results to higher dimensions.

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A End of the proof of Lemma 9

A.1 Case C

We continue here with the third and last case, where two robots enter the triangle T through a point $C^* = p_0$ along its side $[AB]$. The goal is to wake up T in two time units, assuming that the diameter of T is normalized to 1. As previously, the strategy depends on the number of sleeping robots in certain subregions. It also depends on whether the two robots are located along $[AF]$ or $[FB]$. Let P_B denote the parallelogram $BFED$ (i.e., the union of triangles T_B and T_0) and let P_A denote the parallelogram $AEDF$ (union of T_A and T_0). Finally, let $P \in \{P_A, P_B\}$ be the parallelogram that contains p_0 . The main cases are as follows. If P contains no sleeping robots, we apply Case C0. Otherwise, the strategy depends on the number of sleeping robots in the triangle containing p_0 . If it contains none, we apply Case C1; if it contains exactly one, Case 2; and if it contains two or more, Case C3. The cases are illustrated in Figure 7.

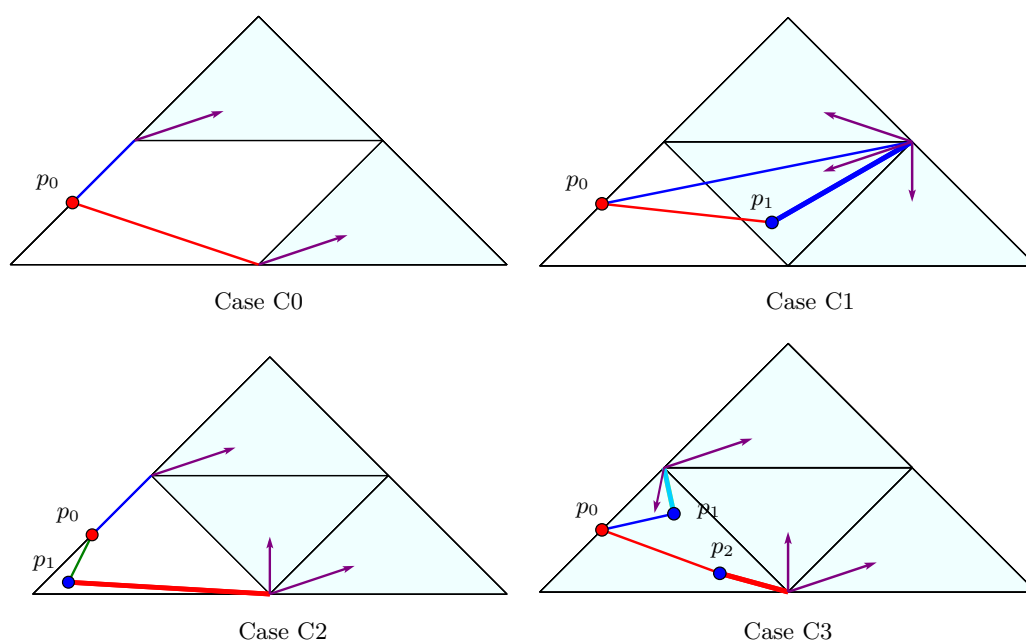


Figure 7 The five subcases of Case C (Lemma 9).

- **Case C0.** P is empty. If $p_0 \in [FB]$, one robot goes to D and wakes up T_C (Case B), the other goes to F and wakes up T_A (Case B). If $p_0 \in [AF]$, one robot goes to E and wakes up T_C (Case A), the other goes to F and wakes up T_B (Case A). It may happen that one of the subtriangles contains the same number of sleeping robots than T itself, but since we recurse in Case A and Case B, the number of robots will inevitably decrease subsequently. The makespan is at most 2.
- **Cases C1.** P is not empty and the triangle containing p_0 is empty. If $p_0 \in [FB]$, one of the robots goes directly to E , the other wakes up p_1 . Then, the two resulting robots in p_1 go to E as well. The path (p_0, p_1, E) can always be realized through two monotonic parts, one in T_B (of length at most $1/2$) and one in T_0 (same), thus it has length at most

one. Then, the three robots wake up T_A, T_0 , and T_C (each of diameter $1/2$) in one time unit by recursing in Case B, Case B, and Case A, respectively. If $p_0 \in [AF]$, one of the starting robots goes to F (to wake up T_B in Case A), the other wakes up p_1 and move with it to D before one time unit overall (by the same arguments). Finally, these two robots wake up T_0 (Case A) and T_C (Case B) in another time unit.

- **Cases C2.** P is not empty and the triangle containing p_0 has exactly one sleeping robot, say at position p_1 . If $p_0 \in [FB]$, one of the two robots goes directly to F . The other wakes up the robot at p_1 and the two resulting robots move to D . The path (p_0, p_1, D) is at most 2-monotonic within T_B of diameter $1/2$, thus these robots arrive at D in at most one time unit. These two robots wake up T_0 (Case A) and T_C (Case B) in another time unit. Similarly, the robot at F wakes up T_A (Case B) in at most one time unit. If $p_0 \in [AF]$, one of the two robots goes directly to E , the other one wakes up p_1 . One of them wakes up T_C and the other goes to F and wakes up T_A . The path (p_0, p_1, F) is at most 2-monotonic in a triangle of diameter $1/2$, thus all the robots are ready to wake up their assigned subtriangle before one time unit.
- **Case C3.** P is not empty and the triangle containing p_0 has at least two sleeping robots, say at positions p_1 and p_2 . If $p_0 \in [FB]$, the two robots wake up (separately) p_1 and p_2 , which gives four awake robots. Two of them move to F , the two others move to D , arriving at these locations before one time unit (2-monotonic paths in a triangle of diameter $1/2$). From these locations, each of the four robots wakes up one of the four subtriangles. If $p_0 \in [AD]$, the strategy is the same, except that two of the four robots move to F and the two others move to E , before waking up (separately) the four subtriangles.

A.2 An illustrative scenario

A more complex wake-up tree is shown on Figure 8, which involves many different cases.⁴ From Theorem 1, we are in the regime $n_0 \geq 11$, and thus we have to recruit to the densest triangle first. Then, seven robots go back to the origin in order to wake up the other seven triangles. In each triangle, Lemma 9 applies. Along the induction, further subtriangles are considered. The makespan may not be optimal, but it is lower than 5 by Theorem 1.

B Detailed proof of Theorem 2

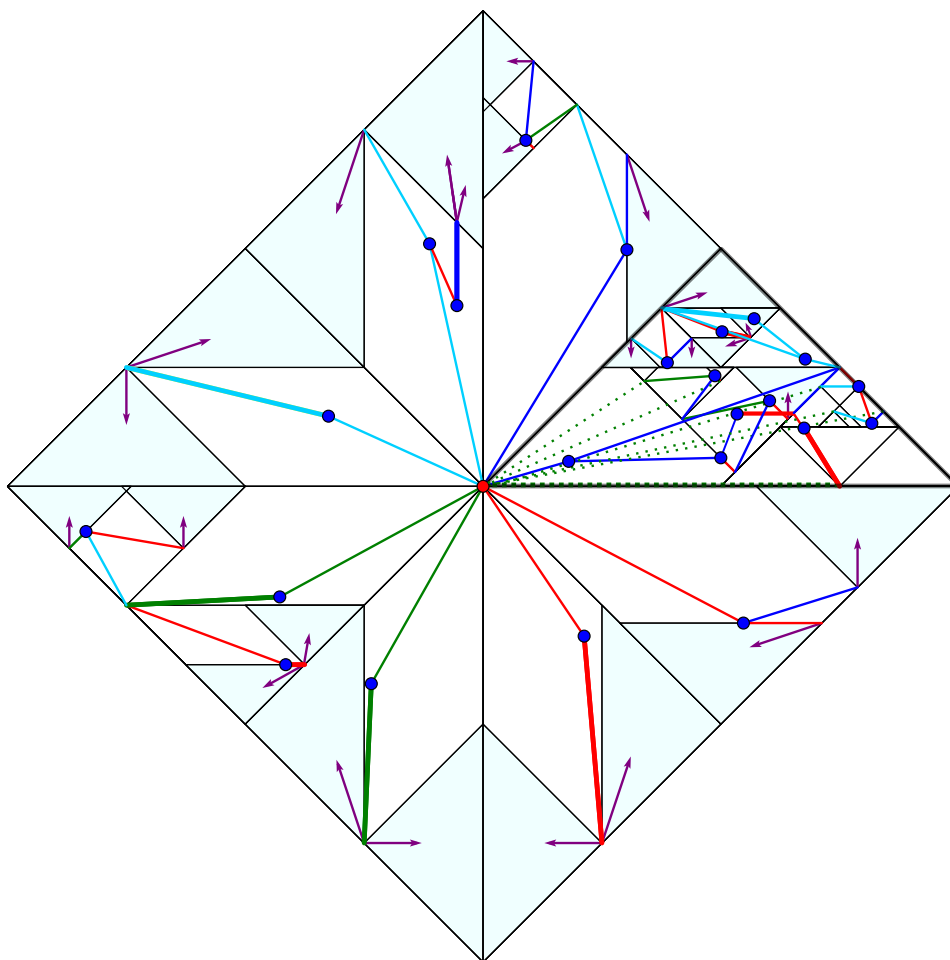
The goal of this section is to prove Theorem 2.

► **Theorem 2.** *Let η be any norm and let $\tau > 3$ be any real such that $\tau \geq \gamma(\eta)$. Knowing τ , one can construct in time $O(n)$ a wake-up tree of makespan at most $\gamma(\eta)$ for any set of n points in the unit η -disk and rooted at the origin.*

For this purpose, let us introduced two simple strategies: **Heap-Strategy** and **Split-Cone-Strategy**. These strategies apply to (\mathbb{R}^2, η) , for any norm η .

The positions of robots are represented by a point set $P = \{p_0, p_1, \dots, p_n\}$ in the unit η -disk, where $p_0 = (0, 0)$ is the position of the awake robot, and p_i is the position of the i th sleeping robot, $i \in \{1, \dots, n\}$.

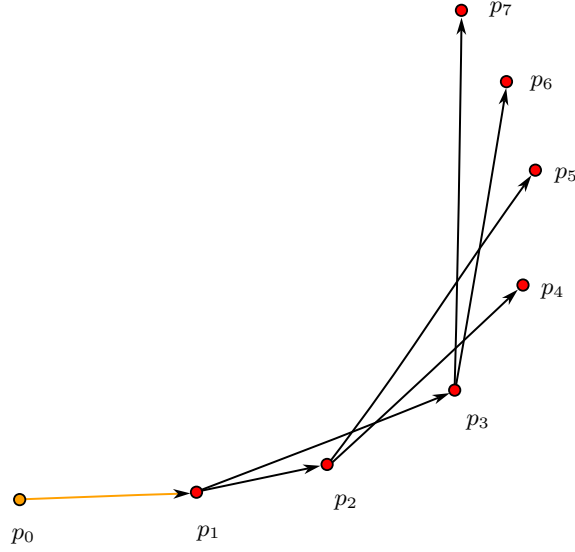
⁴ This construction was computed by an actual implementation of our algorithm.



■ **Figure 8** An illustration of our construction by applying the first steps of the inductive Lemma 9. In these representation, blue triangles correspond to region where sleeping robots are not depicted. An arrow outgoing from a vertex X in a triangle indicates that all the sleeping robots of this one will be woken up by the awake robots at position X . A thick edge indicates that two awake robots follow the same path. The bold triangle is the initial triangle where Lemma 9 is invoked. Dotted edges indicates the move of some robots to the origin. After the robots in the first triangle are awakened, seven awake robots come back to the origin to wake up the seven other triangles (using again Lemma 9).

Heap-Strategy consists in building a minimum heap (binary) tree H for $\{p_1, \dots, p_n\}$ where the key of p_i is the distance from p_0 to p_i , i.e., $\eta(p_0 - p_i)$. The wake-up tree rooted at p_0 is then composed of H itself, plus the edge connecting p_0 to the root of H (its top element), i.e., the closest point from p_0 . Using the well-known “build-heap” and “heapify” routines, H and thus the wake-up tree can be constructed in time $O(n)$.

Heap-Strategy has the interesting property of constructing, in time $O(n)$, a *non-decreasing* wake-up tree for P : each robot is always woken up by a robot which is closer to p_0 than it itself is. In other words, for each edge (p_i, p_j) of the tree, where p_i is the parent of p_j , $\eta((p_0, p_i)) \leq \eta((p_0, p_j))$. See Figure 9. As we will see, this strategy is efficient (it achieves a low makespan) whenever P is contained in a region of small width, e.g., inside a parallelogram whose height is much smaller than its length.



■ **Figure 9** The **Heap-Strategy** applied to p_0, p_1, \dots, p_7 , here in convex position. The resulting wake-up tree has the non-decreasing property (here w.r.t. ℓ_2). Building the tree can be done in time $O(n)$, thus even faster than sorting or computing a convex hull.

This property leads to a first application.

► **Proposition 12.** *If the points of P are on a line, then an optimal wake-up tree for P can be computed in $O(n)$.*

Proof. Let L be the line of (\mathbb{R}^2, η) containing P . By removing p_0 from L split (p_1, \dots, p_n) into two sets: A and B . Let p_a (resp. p_b) be the closest point of A (resp. B) from p_0 . And, let $p_{a'}$ (resp. $p_{b'}$) be the farthest point of A (resp. B) from p_0 .

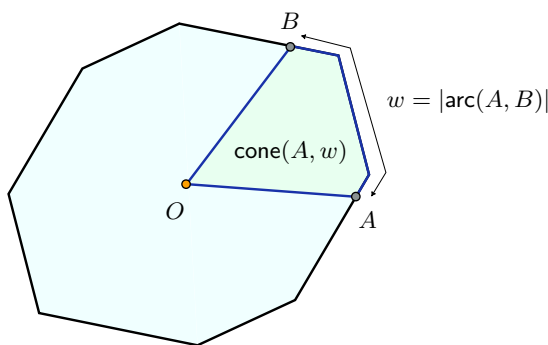
Observe that an optimal wake-up tree can always be transformed into a wake-up tree T with same makespan whose first edge is $p_0 - p_u$ for some $u \in \{a, b\}$. This is because “jumping” over p_a or p_b cannot improve the makespan. Then, from p_u , there must be a branch in T that reach $p_{a'}$ and $p_{b'}$, leading to a branch (from p_u) of length at least $\max\{\eta(p_u - p_{a'}), \eta(p_u - p_{b'})\}$. In other words, the minimum makespan is at least

$$\min_{u \in \{a, b\}} \{\eta(p_0 - p_u) + M(u)\}, \quad \text{where } M(u) = \max_{v \in \{a', b'\}} \{\eta(p_u - p_v)\} \quad (1)$$

We construct a wake-up tree with such makespan using **Heap-Strategy** as follows. Apply **Heap-Strategy** to the subset A , with root p_a , giving a wake-up tree T_A . Since T_A is non-decreasing, and p_a is an endpoint of A , the makespan of T_A is precisely $\eta(p_a - p_{a'})$, $p_{a'}$ being the second endpoints of A . Similarly, applying **Heap-Strategy** to B gives a wake-up tree T_B with root p_b . This takes time $O(n)$. From T_A and T_b , we can construct a first wake-up tree T'_A for P by connecting T_A and T_B with the edges $p_0 - p_a$ and $p_a - p_b$. This gives a valid wake-up tree with makespan $\eta(p_0 - p_a) + \max\{\eta(p_a - p_{a'}), \eta(p_a - p_{b'})\}$, which is exactly $\eta(p_0 - p_a) + M(a)$. Similarly, we can construct a second tree T'_B for P by connecting T_A and T_B with the edges $p_0 - p_b$ and $p_b - p_a$. This gives a valid wake-up tree with makespan $\eta(p_0 - p_b) + M(b)$. By taking the best of T'_A and T'_B , we obtain in time $O(n)$ a wake-up tree of root p_0 of makespan $\min_{u \in \{a, b\}} \{\eta(p_0 - p_u) + M(u)\}$, which is exactly the lower bound in Eq.(1). ◀

Note that **Heap-Strategy** could replace efficiently the **Greedy-Strategy** discussed in [SABM04, KLS05]: nearest sleeping robot is awakened first⁵. The latter runs in time $O(n^{2-2/(\lceil d/2 \rceil + 1) + \varepsilon})$ for (\mathbb{R}^d, ℓ_2) , see [SABM04], thus time $O(n^{1+\varepsilon})$ for $d \in \{1, 2\}$. In fact, even for $d = 1$, **Greedy-Strategy** requires $\Omega(n \log n)$ time for points on a line, by a simple reduction from sorting n numbers. It was proved that, for $d = 1$, the **Greedy-Strategy** leads to a 4-approximation [SABM04, Th. 3] and that the approximation ratio is at least $4 - \varepsilon$ (cf. [SABM04, Th. 1]).

A second, and more important application is when the points of P are in a *cone*. The *unit circle*, w.r.t. the η -norm, is the boundary of the unit disk. Let $\pi(\eta)$ be the *half-circumference* of the unit circle. So, the number $\pi \approx 3.14$ is nothing else than $\pi(\ell_2)$. We know from Gołab's Theorem that $\pi(\eta) \in [3, 4]$, both bounds being attained for affinely regular hexagons and parallelogramms, respectively. (E.g., see [Sch76, Th.4I-4K, pp.27]). Given two points A, B of the unit circle, denote by $\text{arc}(A, B)$ the part of the circle that is traversed anti-clockwise from A to B on the circle. The *length* of $\text{arc}(A, B)$ is $|\text{arc}(A, B)| \in [0, 2\pi(\eta))$, measured in the η -norm. Given a real $w \in [0, 2\pi(\eta))$, and a point A of the unit circle, define $\text{cone}(A, w)$ as the region of the plane composed of all the points of the segments $[OX]$, where X is the point of the unit circle when going anti-clockwise from A and such that $|\text{arc}(A, X)| = w$. The value w is called the *arc-length* of $\text{cone}(A, w)$. If $\eta = \ell_2$, the arc-length of a cone corresponds to its angle. See Figure 10.



■ **Figure 10** The unit η -disk and η -circle for an arbitrary norm η , here given by a symmetric affinely octogon. The region in light-green is $\text{cone}(A, w)$, with arc-length $w = |\text{arc}(A, B)| \in [0, 2\pi(\eta))$. Note that in general, two cones with same arc-length w , say $\text{cone}(A, w)$ and $\text{cone}(A', w)$, cannot be obtained from each other by a rotation around O .

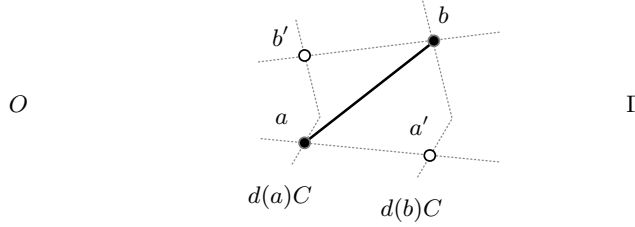
We have:

► **Proposition 13.** *If P is contained in a cone of arc-length w , then **Heap-Strategy** constructs in time $O(n)$ a wake-up tree for P , rooted at the origin, with makespan at most $1 + w \lfloor \log_2 n \rfloor$.*

Proof. Assume $P \subset \text{cone}(X, w)$ for some X on the unit circle, and let T be the wake-up tree produced by **Heap-Strategy**. Let Γ be the arc of length w from X , i.e., the intersection of $\text{cone}(X, w)$ with the unit circle. Denote by $\lambda\Gamma = \{\lambda u \in \mathbb{R}^2 : u \in \Gamma\}$ the arc Γ scaled down by a factor $\lambda \in [0, 1]$. Let $d(p) = \eta(p_0 - p)$. By definition of the norm, every point p on the arc of q satisfies $d(p) = d(q)$.

⁵ There are variants that depend on how conflicts between robot are resolved.

Now, consider any edge $a - b$ of T , where a is the parent of b . The homothetic arc of Γ containing a is $d(a)\Gamma$, whereas the arc containing b is $d(b)\Gamma$. Let $a' = [Oa] \cap d(b)\Gamma$, the projection of a along the segment $[Oa]$ on the arc of b . Similarly, let $b' = [Ob] \cap d(a)\Gamma$. See Figure 11.



■ **Figure 11** Bounding the length of an edge $a - b$ of T constructed by **Heap-Strategy** for points in a cone (not represented) with apex O and arc-length $|\Gamma|$.

W.l.o.g. assume $a \in d(a)\text{arc}(X, b')$. The other case, $b' \in d(a)\text{arc}(X, a)$ is similar. By the triangle inequality, we can bound the length of the edge $a - b$ by the length of the path $a - a' - b$. The latter is at most $|d(a') - d(a)| + |\text{arc}(a', b)|$. We have $|\text{arc}(a', b)| \leq w$, and $d(a') = d(b)$ because a' belongs to the arc of b . Moreover, since T is non-decreasing, $d(a) \leq d(b)$. Therefore, the length of $a - b$ is $|ab| \leq d(b) - d(a) + w$.

Consider any branch $(p_0, a_0, a_1, \dots, a_h)$ of T . Its total length is bounded by:

$$d(a_0) + \sum_{i=1}^h (d(a_i) - d(a_{i-1}) + w) = d(a_h) + wh \leq 1 + w \lfloor \log_2 n \rfloor .$$

Indeed, clearly, $d(a_k) \leq 1$, and h is the number of edges in the branch rooted at a_0 . We conclude by the fact that a_0 and its descendants form a binary heap on n elements, and thus of depth at most $\lfloor \log_2 n \rfloor$. ◀

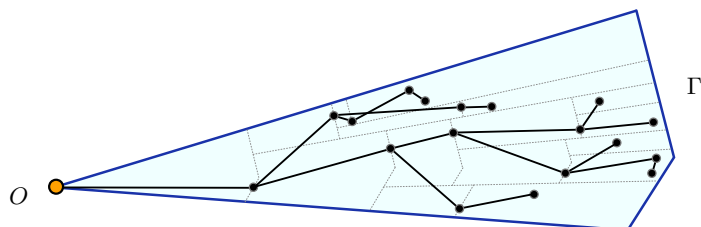
In term of makespan, **Heap-Strategy** is not optimal because it produces in the wake-up tree branches that may zigzag in the cone, each turn having possibly a cost of w in the worst case. This can be corrected using the **Split-Cone-Strategy**. Roughly speaking, the strategy constructs again a non-decreasing tree with the extra property that each subtree, after the i first steps, wakes up subcones whose arc-length becomes exponentially smaller. This involves the golden ratio $\varphi = (1 + \sqrt{5})/2 \approx 1.61$.

► **Proposition 14.** *If P is contained in a cone of arc-length w , then **Split-Cone-Strategy** constructs in time $O(n \log n)$ a wake-up tree for P , rooted at the origin, of makespan at most $1 + \varphi w$.*

Proof. Assume $P \subset \text{cone}(X, w)$ for some X on the unit circle, and let Γ be the arc of length w from X . Let $c = \varphi - 1 = 1/\varphi \approx 0.61$, where $\varphi = (1 + \sqrt{5})/2$ is the golden ratio.

The wake-up tree for P is constructed as follows (Figure 12). As for **Heap-Strategy**, the first edge connect p_0 to its closest (w.r.t. η -norm) sleeping robot at a position, say $a \in d(a)C$, where $d(a) = \eta(p_0 - a)$. We then split the current $\text{cone}(X, w)$ into two subcones C and C' defined as follows: C contains a and its arc-length is cw , whereas C' of arc-length $(1 - c)w$ is the complementary cone of C in $\text{cone}(X, w)$. Then, from a , the wake-up tree continues in parallel in C and in C' , and connects a with the closest point $b \in C$ and the closest point $b' \in C'$. The process continues recursively from b within the subcone C , and from b' within

the subcone C' , according to the same rule of splitting: the current subcone C or C' being subdivided according to the ratio c or $1 - c$, the fraction c containing the current point. We repeat the process until all the points have been spanned.



■ **Figure 12** Illustration of the **Split-Cone-Strategy**, producing non-decreasing wake-up trees. Each arc homothetic to Γ going thru a point p_i is split into two sub-arcs, and defines two subcones: one of arc-length $(1/\varphi)|\Gamma| \approx 0.61|\Gamma|$ (containing p_i) and its complementary of arc-length $\approx 0.39|\Gamma|$.

It is easy to check that the corresponding tree T is non-decreasing and can be computed in time $O(n \log n)$, as it requires to sort all the points according to the η -distance from p_0 .

It remains to analyze the makespan of T . Consider an edge (a, b) of T , a the parent of b . Using the triangle inequality (as we did in the proof of Proposition 13), we can bound the length of (a, b) by a contribution on the segment $[OX]$ and a contribution on the arc-lengths. Due to a telescopic sum (T is non-decreasing), the total contribution on the segment $[OX]$ sums up to at most 1. For the contribution in the arc-length, we can proceed by induction. Assume that a is in a subcone of arc-length $x \leq w$, and denote by $f(x)$ the maximum arc-length contribution for a branch starting from a to any leaf of T .

Let us show that the arc-length contribution $f(x)$ fulfills the equation:

$$f(x) \leq \max \{cx + f(cx), x + f((1 - c)x)\} . \quad (2)$$

Indeed, there are two cases.

- If $b \in C$, i.e., b belongs to the same subcone of a , then the arc-length contribution for (a, b) is at most cx plus a contribution for any branch starting from b to a leaf of T , which is by induction $f(cx)$ since all the descendents of b in T will be in C , a subcone of arc-length cx .
- If $b \in C'$, i.e., b belongs to the complement subcone of a with arc-length $(1 - c)x$, then the arc-length for (a, b) is at most x plus the contribution for any branch starting from b which is $f((1 - c)x)$ since b belongs to C' .

By induction and by plugging $c = 1/\varphi$, it is not difficult to check that $f(x) < \varphi x$. Indeed, the first term of Eq.(2) gives, $cx + f(cx) < x(1/\varphi + 1) = \varphi x$. And, the second term gives, $x + f((1 - c)x) < x(1 + (1 - 1/\varphi)) = \varphi x$. So, both terms are satisfied.

The makespan of the **Split-Cone-Strategy** is therefore at most $1 + f(w) < 1 + \varphi w$. ◀

The **Split-Cone-Strategy** has an interesting corollary:

► **Proposition 15.** For any norm η , and every $n \in \mathbb{N}$,

$$\gamma_n(\eta) < 3 + \frac{4\varphi\pi(\eta)}{|1 + \sqrt{1+n}|} \quad \text{and} \quad \gamma(\eta) < 3 + \varphi\pi(\eta) \leq 3 + 4\varphi .$$

Proof. Let k be the least integer such that $k(k-2) \geq n$. Since $k(k-2) = (k-1)^2 - 1$, this integer is $k = \lceil 1 + \sqrt{n+1} \rceil$.

In order to construct a low makespan wake-up tree, we shall use twice the **Split-Cone-Strategy** as follows. We split the unit disk into k equal cones, so each with arc-length $w = 2\pi(\eta)/k$. At a first phase, we construct a wake-up tree in the densest cone by applying the **Split-Cone-Strategy**.

From Proposition 14, we obtain a wake-up tree of makespan at most $1 + \varphi w$. Because $k(k-2) \geq n$, the densest cone (among k) contains at least $k-2$ sleeping robots. When all of them are awakened, we have, with p_0 , a total of $k-1$ awake robots (at least). For the second phase, we construct in parallel wake-up trees for the $k-1$ remaining cones thanks again to the **Split-Cone-Strategy**. Combining trees is possible, the number of awake robots contained in the wake-up tree⁶ during the first phase being at least the number of trees in the second phase. So, we can connect them.

This leads to a wake-up tree of makespan less than $(1 + \varphi w) + 1 + (1 + \varphi w) = 3 + 2\varphi w$. Plugging the values of w and k , we get the claimed upper bound for $\gamma_n(\eta)$.

To prove the second inequality, we construct a wake-up tree thanks to the following strategy: (1) wake-up any robot, and come back to the origin with two awake robots; and (2) wake-up in parallel each of the half-disk, that is a cone of arc-length $\pi(\eta)$, using the **Split-Cone-Strategy**. The resulting makespan is less than $3 + \varphi\pi(\eta)$.

The last inequality comes from the fact that $\pi(\eta) \leq 4$, for every norm η . ◀

One can check for ℓ_2 -norm, by plugging $n = 528$ and $\pi(\ell_2) = \pi$ in the equation of Proposition 15, that $\gamma_n(\ell_2) < 1 + 2\sqrt{2}$. Therefore, Conjecture 6 – *It's always quicker to wake up n robots than four* – is confirmed for $n \geq 528$.

The drawback of **Split-Cone-Strategy** is that its construction does not take a linear time. However, we can combined both strategies to get (almost) the best of the both strategies. The resulting strategy, described in the proof of Proposition 16, is called **Linear-Split-Strategy**.

► **Proposition 16.** *If P is contained in a cone of arc-length w , the **Linear-Split-Strategy** constructs in time $O(n)$ a wake-up tree for P , rooted at the origin, with makespan at most $1 + \varphi w + o(w/n^{2/3})$.*

Proof. Assume $P \subset \text{cone}(X, w)$ for some X on the unit circle. Let $K = \lceil n/\log_2 n \rceil$, and define $A \subset P$ composed of the K closest points from p_0 , breaking tie arbitrarily, and let $B = P \setminus A$. Using heap-sort, one can construct A (and B) in time $O(n + K \log n) = O(n)$.

We apply, the **Split-Cone-Strategy** on A that, from Proposition 14, produces in time $O(K \log K) = O(n)$ a wake-up tree T_A . We also subdivide $\text{cone}(X, w)$ into $\lceil w/s \rceil$ consecutive subcones, each of arc-length s , a number that will be fixed later. Let C_i be the i th such subcones, $i \in \{1, \dots, \lceil w/s \rceil\}$. We compute the sets $B_i \subset B \cap C_i$, the set of points of B that fall into the subcone C_i (breaking tie arbitrarily). Note that some B_i may be empty. We apply the **Heap-Strategy**, independently for each $\{p_0\} \cup B_i$, set that is contained in C_i (if not empty). This produces at most $\lceil w/s \rceil$ wake-up trees, one tree T_{B_i} for each $\{p_0\} \cup B_i$. From Proposition 13, all the trees T_{B_i} can be constructed in time $\sum_{i=1}^{\lceil w/s \rceil} O(|B_i|) = O(w/s) + O(\sum_i |B_i|) = O(w/s + n)$. We will require that $w/s = O(n)$.

⁶ This number is precisely twice the number of leaves plus the number of vertices with one child.

It remains to combine T_A and T_{B_i} trees. For that, we update each tree T_{B_i} by removing its root p_0 . Now, the arc-length s is chosen large enough such that, if $B_i \neq \emptyset$, then at least one leaf of T_A falls into C_i . Then, for each tree T_{B_i} , we connect its new root (the closest point of B_i from p_0 , since p_0 is not anymore in T_{B_i}) to any leaf of T_A that belongs to subcone C_i . This leads to the willing wake-up tree T for P .

From the analysis of the **Split-Cone-Strategy** in the proof of Proposition 14, an edge of T_A of depth k leads to subcones of arc-length at most $t = \max_{i+j=k} (1/\varphi)^i (1 - 1/\varphi)^j w$. This is because at each edge, either the arc-length of the current cones is multiplied by a factor $(1/\varphi)$ or $(1 - 1/\varphi)$. Because $1/\varphi > 1 - 1/\varphi$, it follows that $t = w/\varphi^k$. The depth of T_A , that spans K points, is at least $\lceil \log_2 K \rceil$. So, the subcones of maximal depth and containing any leaf of T_A are of arc-length most $w/\varphi^{\lceil \log_2 K \rceil} \leq 2w/K^{\log_2 \varphi}$. By choosing $s = 4w/K^{\log_2 \varphi}$ (so twice larger), we ensure that the final subcones of maximal depth and containing any leaf of T_A is contained in some subcones C_i of arc-length s . We check also that $w/s < K^{\log_2 \varphi} < K = O(n)$ as required.

It remains to bound the makespan of T . Note that T is non-decreasing. Therefore, the radius contribution of any branch is at most 1. For the arc-contribution, this is at most φw for T_A , and then at most $s \lceil \log_2 n \rceil$ for T_{B_i} (by Proposition 13 and Proposition 14). In total, the makespan is at most

$$1 + s \lceil \log_2 n \rceil \leq 1 + w\varphi + \frac{4w}{K^{\log_2 \varphi}} \lceil \log_2 n \rceil \leq 1 + w\varphi + 4w \frac{\lceil \log_2 n \rceil}{\lceil n/\log_2 n \rceil^{\log_2 \varphi}} \quad (3)$$

$$\leq 1 + w\varphi + o\left(\frac{w}{n^{2/3}}\right) \quad (4)$$

noting that $\log_2 \varphi \approx 0.69 > 2/3$. ◀

We are now ready to proof Theorem 2. Let us recall its statement.

▶ **Theorem 2.** *Let η be any norm and let $\tau > 3$ be any real such that $\tau \geq \gamma(\eta)$. Knowing τ , one can construct in time $O(n)$ a wake-up tree of makespan at most $\gamma(\eta)$ for any set of n points in the unit η -disk and rooted at the origin.*

Proof. Similarly to Proposition 15, we can construct in time $O(n)$ a wake-up tree for P with makespan at most $3 + c/\sqrt{n}$ for some constant c large enough. Indeed, one can split the unit disk into \sqrt{n} cones, each of arc-length $w = 2\pi(\eta)/\sqrt{n}$, and wake up the densest one. Then, using the **Linear-Split-Strategy** (Proposition 16) in this cone, containing $m \geq \sqrt{n}$ points, we can wake up m robots with a makespan (remember that $\pi(\eta) \leq 4$):

$$\begin{aligned} 1 + \varphi w + o(w/m^{2/3}) &= 1 + \varphi w + o\left(\frac{2\pi(\eta)}{\sqrt{n}} / m^{2/3}\right) \\ &= 1 + \varphi w + o(n^{-5/6}). \end{aligned}$$

Coming back to the origin, and repeating in parallel the **Linear-Split-Strategy** for all cones with some sleeping robots (at most \sqrt{n} cones), we can complete the waking up. The time to build all these trees is $\sum_{i=1}^{\sqrt{n}} O(n_i) = O(\sqrt{n}) + O(\sum_i n_i) = O(n)$, where n_i is the number of sleeping robots in the i th cone. The makespan of the construction is

$$3 + 2\varphi w + o(n^{-5/6}) < 3 + \frac{26}{\sqrt{n}} + o(n^{-5/6}) \leq 3 + \frac{c}{\sqrt{n}} \quad (5)$$

for a constant $c > 26$ large enough (using the facts that $\pi(\eta) \leq 4$ and that $8\varphi < 13$). Actually, c can be precisely determined from Eq.(3) in the proof of Proposition 16. The lowest order

term in Eq.(3) is $\leq 4w$, for a single application of **Linear-Split-Strategy**. So, after two applications of the strategy, and plugging $w = 2\pi(\eta)/\sqrt{n} \leq 8/\sqrt{n}$, we get a makespan of $3 + 26/\sqrt{n} + 8w \leq 3 + (26 + 64)/\sqrt{n}$. Thus, $c \leq 90$ is enough.

Now, assume that $\tau > 3$ and $\tau \geq \gamma(\eta)$. Compute the least integer $n_0 \geq (c/(\tau - 3))^2$. Note that n_0 is a fixed constant, independent of n .

- If $|P| = n \geq n_0$, then we can apply the previous strategy providing a makespan that is less than $3 + c/\sqrt{n} \leq 3 + c/\sqrt{n_0} \leq \tau$ by Eq.(5) and by the choice of n_0 .
- If $|P| = n < n_0$, then we can brute force for finding an optimal wake-up tree whose makespan is at most $\gamma(\eta)$ by definition of $\gamma(\eta)$. This is also at most τ by the choice of τ . The number of wake-up trees we have to consider in a brute force algorithm is at most $n! \leq n_0! = O(1)$, and checking the makespan of each of these trees costs $O(n) = O(1)$.

In both cases, we have constructed a wake-up tree in time $O(n)$ and with makespan $\leq \tau$ as required. This completes the proof. \blacktriangleleft

We note that Eq.(5) in the proof of Theorem 2 implies an $(3 + o(1))$ -approximation running in time $O(n)$. A similar result was already proved in [ABG⁺03, Th. 1]. However, our construction, based on cones, gives a better second order term, namely $O(1/\sqrt{n})$, whereas the $o(1)$ term given in the proof of [ABG⁺03, Th. 1] is $\Omega(\log n/n^{1/4})$.

C Experiments

We have done some experiments, and we have computed numerically, by a brute force algorithm⁷ the minimum makespan for points that are equally distributed on the unit circle. Table 1 shows the results for ℓ_2 -norm, but results for other norms are available. We observe that, for this distribution, the optimal makespan denoted by $\text{unif}(n)$ are essentially decreasing with n , for a given parity and $n \geq 4$, with some exceptional cases.

The exceptional cases imply that one the two following quite reasonable statements is wrong: (1) the wake-up ratio is attained for points that are equally distributed on the unit circle; (2) for every $n \geq 4$, $\gamma_n(\ell_2) > \gamma_{n+2}(\ell_2)$.

D The Exact Value of $\gamma_4(\eta)$

► **Theorem 3.** *For any norm η , $\gamma_4(\eta) = 1 + \Lambda(\eta)$.*

Proof. Consider a set of four points, $X = \{A, B, C, D\}$ (the sleeping robots), taken in the unit η -disk, and let $O = (0, 0)$ be the origin, where the awake robot is placed.

Lower bound. To show that $\gamma_4(\eta) \geq 1 + \Lambda(\eta)$, assume that X forms the largest parallelogram inscribed in the unit η -disk. Note that points are on the boundary of the unit disk. Any wake-up tree rooted at O and spanning $X \cup \{O\}$ must have a branch with at least three edges, say e_1, e_2, e_3 . The first edge e_1 has length $\eta(e_1) = 1$ since all points of X are on the boundary of the unit disk. The next two edges e_2, e_3 must be taken among the $\binom{|X|}{2} = 6$ segments of X (defined by any pair of points in X), namely $e_1, e_2 \in \{s_1, s_2, s_3, \bar{s}_1, \bar{s}_2, \bar{s}_3\}$,

⁷ Code available on demand to the authors.

n	$\text{unif}(n)$
4	3.828
5	3.351
6	3.732
7	3.431
8	3.613
9	3.416
10	3.520
11	3.383
12	3.449
13	3.349
14	3.454
15	3.318
16	3.443
17	3.331

■ **Table 1** Optimal makespan $\text{unif}(n)$ (numerical approximation) for n points equally distributed on the unit disk, with ℓ_2 -norm. Boxed values are exceptional cases such that $\text{unif}(n) > \text{unif}(n-2)$ and $n \geq 4$.

where $(s_1, s_2, \bar{s}_1, \bar{s}_2)$ correspond to the four consecutive sides of the boundary of X , and s_3, \bar{s}_3 correspond to the two diagonals of X . Because e_2 and e_3 must be consecutive segments of X (say $e_2 = (A, B)$ and $e_3 = (B, C)$ for instance), we have $e_2 \in \{s_i, \bar{s}_i\}$ and $e_3 \in \{s_j, \bar{s}_j\}$ for some $i \neq j \in \{1, 2, 3\}$. Clearly, $\eta(s_i) = \eta(\bar{s}_i)$ and $\eta(s_i) \leq \eta(s_3) = 2$. Because we want to lower bound $\eta(e_2) + \eta(e_3)$, we can assume that $i, j < 3$, i.e., $i = 1$ and $j = 2$ or the reverse. We conclude with the fact that $\eta(s_1) + \eta(s_2) = \Lambda(\eta)$, and thus $\eta(e_1) + \eta(e_2) + \eta(e_3) \geq 1 + \Lambda(\eta)$.

Upper bound. It remains to prove $\gamma_4(\eta) \leq 1 + \Lambda(\eta)$.

We will use the following facts.

► **Fact 1.** *If $C_1 \subset C_2$ are two convexes, then the perimeter of the boundary of C_1 is less than the perimeter of the boundary of C_2 (see [Sch76, Th. 4C p. 25] for instance).*

► **Fact 2.** *Any quadrilateral contained in the unit η -disk has half-perimeter at most $\Lambda(\eta)$.*

This latter fact is a consequence of Fact 1 and of the central symmetry of unit disk.

It is well-known that in any set of five points contains four points in convex position (see [MS00]). Note that in our setting, four points do not determine necessarily a quadrilateral since points are not necessarily in general position (and so some side may contain more than two points). Since we are concerned with the perimeter, for convenience, we will still call it a quadrilateral whereas we should speak about the four points on its convex hull.

Let Q be a subset of X forming a quadrilateral, that is a convex having four points of its convex hull. There are two cases.

Case 1. $O \notin Q$. In that case, we use the “racquet” strategy: O goes to any point of Q (at cost at most 1); then in parallel, one robot turns clockwise and the other one anti-clockwise around the convex hull of Q with an extra cost of half the perimeter of quadrilateral Q . Overall the cost is at most $1 + \Lambda(\eta)$ from Fact 2.

Case 2. $O \in Q$. W.l.o.g. assume that Q is $OABC$ in this order. We have $D \notin Q$. Denote by $\bar{A}, \bar{B}, \bar{C}$ be the opposite points of A, B, C respectively, the symmetric points around O . There are two subcases.

Case 2a. C belongs to the convex hull of A, B, C, \bar{A} . In that case the robot in O goes to A (at cost at most 1); then in parallel one robot goes to D (with extra cost of 2), while the other goes to B and then to C . The branch (O, A, D) has length at most $3 \leq 1 + \Lambda(\eta)$ since $\Lambda(\eta) \geq 2$. One can upper bound the length of the path (A, B, C) by $\Lambda(\eta)$. Indeed, we observe that, by translating the triangle $\bar{A}\bar{B}\bar{C}$ by a vector $A - \bar{C}$, one can form a parallelogram $ABC\bar{B}$ that is contained in the unit disk (because it is included in the hexagon $ABC\bar{A}\bar{B}\bar{C}$). It follows that the length of the path (A, B, C) , i.e., $\eta((A, B)) + \eta((B, C))$, is at most $\Lambda(\eta)$. It follows that the length of the branch (O, A, B, C) is at most $1 + \Lambda(\eta)$.

Case 2b. C does not belong to the convex hull of A, B, C, \bar{A} . It follows that C is inside the triangle $AB\bar{A}$. In that case, the robot in O goes to C ; then in parallel one robot goes to D (with extra cost of 2), while the other goes to B and then to A . The branch (O, C, D) has length at most $3 \leq 1 + \Lambda(\eta)$ since $\Lambda(\eta) \geq 2$. Remains to bound the length of the branch (O, C, B, A) . We first observe that C is inside the subtriangle $OB\bar{A}$. Indeed, C cannot be inside the subtriangle OAB since Q is $OABC$ that is convex under Case 2 hypothesis. The branch (O, C, B, A) has a length that is bounded by the length of (O, \bar{A}, B, A) . Indeed, by Fact 1, the triangle OCB has perimeter no more than the perimeter of the triangle $O\bar{A}B$. We conclude with the fact that $AB\bar{A}\bar{B}$ is a parallelogram contained in the unit disk. Therefore, (\bar{A}, B, A) has length at most $\Lambda(\eta)$. It follows that the length of the branch (O, C, B, A) is at most the length of (O, \bar{A}, B, A) that is at most $\eta((\bar{A}, O))$ plus the length of (\bar{A}, B, A) , that is at most $1 + \Lambda(\eta)$. ◀

E Proofs of Corollary 5

► **Corollary 5.** For every $p \in [1, \infty]$, $1 + 2^{1+\max(1/p, 1-1/p)} \leq \gamma(\ell_p) \leq 5 \cdot 2^{\min(1/p, 1-1/p)}$.

Proof. The lower bound is a simple consequence of Theorem 3 and of the fact that $\Lambda(\ell_p) = 2^{1+\max(1/p, 1-1/p)}$.

For the upper bound, we use the inclusion of unit ℓ_p -disk into ℓ_∞ -disk, showing the well-known inequality $\ell_p(u) \leq \ell_\infty(u) \cdot 2^{1/p}$, for every $u \in \mathbb{R}^2$. Moreover, by scaling and the inclusion of unit ℓ_p -disk into ℓ_1 -disk, we have that $\ell_p(u) \leq \ell_1(u) \cdot 2^{1-1/p}$. It follows that

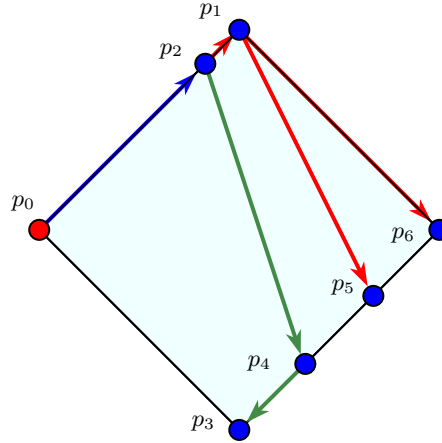
$$\forall p \in [1, \infty], \quad \gamma(\ell_p) \leq \min\left(\gamma(\ell_1) \cdot 2^{1-1/p}, \gamma(\ell_\infty) \cdot 2^{1/p}\right). \quad (6)$$

We conclude with the fact that unit ℓ_1 -disk and unit ℓ_∞ -disk have the same shape under rotation. So, we must have $\gamma(\ell_1) = \gamma(\ell_\infty)$, which is 5 by Theorem 1. The final, upper bound follows from Eq.(6). ◀

F Tightness of Lemma 7

► **Proposition 17.** There are six sleeping robots in a square of diameter 1 that requires a wake-up tree of makespan of at least $13/6$, if rooted at a corner.

Proof. Let $ABCD$ be the vertices of the square of diameter 1, where $A = (0, 0)$, $B = (1/2, 1/2)$, $C = (1, 0)$ and $D = (1/2, -1/2)$. The awake robot, the root, is placed at $p_0 = A$. Then, $p_2 = B$ and p_1 is at distance ε from B for some $\varepsilon \in [0, 1/6]$. The four points p_3, \dots, p_6 are located on $[CD]$ such that they are pairwise at distance at least 2ε . This is possible if $\varepsilon \leq 1/6$. See Figure 13. Observe that the distance between $[AB]$ and $[CD]$ is 1.



■ **Figure 13** One optimal wake-up tree of makespan $13/6$ for $n = 6$ sleeping robots in a square of diameter 1.

Consider any wake-up tree T for p_1, \dots, p_6 rooted at p_0 . There are two cases:

- The first edge of T starts in waking up some robots in $\{p_1, p_2\}$ before going to $\{p_3, \dots, p_6\}$. Then, after a time at least $1 - \varepsilon$, at most three robots are awake before going to $[CD]$ which contains four sleeping robots. Therefore, one of these sleeping robot will be wake up after an extra time of $1 + 2\varepsilon$ since two robots of $[CD]$ are at distance at least 2ε . This gives a makespan for T of at least $(1 - \varepsilon) + (1 + 2\varepsilon) = 2 + \varepsilon$.
- The first edge of T starts in waking up some robots in $\{p_3, \dots, p_6\}$ before going to $\{p_1, p_2\}$. It follows that either p_1 or p_2 is wake up after time $2 + \varepsilon$. Indeed, if p_1 and p_2 are woken up after a time $< 2 + \varepsilon$, then T must have the branches (p_0, p_i, p_1) and (p_0, p_i, p_2) . And, then one robot $p_k \in [CD]$, $k \neq i$, cannot be woken up in time better than $3 > 2 + \varepsilon$.

Overall, the makespan is at least $2 + \varepsilon$ that is $13/6$ if $\varepsilon = 1/6$. ◀