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# Object Location Using Path Separators

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## Abstract

We propose a general approach to solve several object location problems in a large family of weighted graphs, namely in  $k$ -path separable graphs. Our main result is a separator theorem for graphs excluding a fixed minor that separates a graph using  $k$ -paths. Our proof uses the deep structure theorem of Robertson and Seymour. Given any  $k$ -path separable graph, we construct in polynomial time (1) a small-worldization with an average poly-logarithmic number of hops; (2) an  $(1 + \varepsilon)$ -approximate distance labeling scheme with  $O(\log n)$  space labels; (3) a stretch- $(1 + \varepsilon)$  compact routing scheme with tables of poly-logarithmic space; (4) an  $(1 + \varepsilon)$ -approximate distance oracle with  $O(n \log n)$  space and  $O(\log n)$  query time. Such results apply, for instance, to planar graphs, bounded genus graphs, or bounded treewidth graphs. Actually our technique generalizes to bounded-dimension isometric separators, and similar results apply to much wider classes of weighted graphs.

## 1 Introduction

Divide-and-conquer is a widely used paradigm in Computer Science. A celebrated example is the Lipton and Tarjan  $O(\sqrt{n})$ -separator for planar graphs [LT79].

A graph  $H$  is a *minor* of  $G$  if  $H$  is a subgraph of a graph obtained by a series of edge contractions<sup>1</sup> of  $G$ . The study of graphs excluding a fixed minor has led to fundamental graph theory results. In the context of routing, decentralized search, and object location, several natural classes of networks can be defined by their forbidden minor. Among them are *trees* [FG01, Lai04] (excluding  $K_3$ ) and *series parallel* networks [FL03] (excluding  $K_4$ ) that capture many network backbone structures, and *planar graphs* [GKR05] (excluding  $K_5$  and  $K_{3,3}$ ) that capture the structure of two dimensional maps.

Most relevant to our work is that of Thorup [Tho04]. Motivated by the problem of reachability in directed graphs, the author studies object location problems, namely reachability and  $1 + \varepsilon$  distance oracles,  $1 + \varepsilon$  distance labels, and stretch- $(1 + \varepsilon)$  labeled routing schemes, for planar graphs (both directed and undirected). His  $1 + \varepsilon$  distance oracles require  $O(1/\varepsilon \cdot n \log n)$  space and can be distributed into  $O(1/\varepsilon \cdot \log n)$  space  $1 + \varepsilon$  distance labels. The labels can be transformed into a labeled routing scheme.

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<sup>1</sup>The *contraction* of the edge  $e$  with endpoints  $u, v$  is the replacement of  $u$  and  $v$  with a single vertex whose incident edges are the edges other than  $e$  that were incident to  $u$  or  $v$  [Wes01].

Motivated by the "small world" phenomena in social networks, Kleinberg [Kle00], suggested a new probabilistic network model. Specifically, Kleinberg studies the random graph obtained by taking a two-dimensional grid and augmenting it by adding for each node a random edge. This model lead to new algorithmic and graph theoretic questions. One such question studies the complexity of greedy routing on these random graphs. Kleinberg gives a certain distribution of long range contacts that augments the grid and obtains  $O(\log^2 n)$  expected greedy hop count. Fraigniaud [Fra05] and Duchon et al. [DHLS04, DHLS06] suggest to consider the generalized question of finding distributions that augment larger families of graphs and obtain poly-logarithmic expected greedy hop count. Specifically, Fraigniaud obtains  $O(k \log^2 n)$  complexity for graphs with treewidth  $k$  and Duchon et al. obtain poly-logarithmic complexity for growth bounded graphs. Slivkins [Sli05], considers graphs with doubling dimension  $\alpha$  and obtains  $O(2^{O(\alpha)} \log \Delta \log n)$  expected greedy hop complexity. Obtaining poly-logarithmic results for larger families of graphs is an open question recently highlighted by Kleinberg [Kle06]. Very recently, Fraigniaud, Lebhar, and Lotker [FLL06] show that obtaining poly-logarithmic expected greedy hop complexity by augmenting with one random link is impossible for general graphs.

## 1.1 Related work

Finding small separators is an active field of research: in general graphs, the  $O(\sqrt{\log k})$ -approximation for treewidth- $k$  graphs [FHL05], and for planar [LT79, AST94, DV97] (see [AFN03, Table 1 p. 813] for the best current bounds), and more generally for graphs excluding a fixed minor [AST90, DH05, Gro03, PRS94, RW05]. See also [Fu06] for separators in geometric objects. However, all these graphs may have separator of large size, namely of size  $\Omega(\sqrt{n})$ , for a regular  $n$ -vertex mesh for instance, whereas object location problems in regular meshes are obvious.

Object location solutions were studied originally by Awerbuch and Peleg for arbitrary graphs [AP90, AP92]. The general approach was based on sparse covers. However, for every stretch  $s < 3$ , there are unweighted  $n$ -vertex graphs for which every stretch- $s$  routing scheme requires  $\Omega(n)$ -bit labels [GG01], or for which every stretch- $s$  distance labeling scheme requires  $\Omega(n)$ -bit labels [GPPR04, TZ05]. Upper bounds can be greatly improved to  $1 + \varepsilon$  stretch for any  $\varepsilon > 0$  for graphs whose induced metric space has constant doubling dimension [CGMZ05, Sli05, AGGM06]. However even binary rees have unbounded doubling dimension. Moreover the techniques used in all the above papers are not separator based.

Recently [AGM05a] proposed a poly-log memory routing scheme for graphs excluding a minor. The scheme is name-independent, has constant stretch, and is limited to unweighted graphs. This limitation is inherent to name-independent schemes since there is a polynomial space lower bound for name-independent routing for weighted trees [AGM05b]. We stress that the stretch of the name-independent scheme is at least 3 and depends on the excluded minor, so it cannot be fixed arbitrary close to 1.

For graphs excluding a fixed minor, the structure theorem of Robertson and Seymour [RS86, RS03] has already several important algorithmic applications by Demaine et al. [DFHT05, DH05]. Specifically, they develop subexponential fixed-parameter algorithms for dominating set, vertex cover, and set cover in any class of graphs excluding a fixed minor.

## 1.2 Our contributions

Our algorithmic results are a non-trivial generalization of [Tho04] to all graphs excluding a fixed minor. This solves an open question raised by Thorup [Tho04]. These results are based on a new separator theorem whose proof uses the structure theorem of Robertson and Seymour [RS86, RS03]. Our small word result significantly extends the result of [Fra05] to weighted planar graphs and, in general, to all weighted graphs excluding a fixed minor.

The following new definition is central to our paper.

**Definition 1** *A weighted graph  $G$  with  $n$  vertices is  $k$ -path separable if there exists a subset of vertices  $S$ , call  $k$ -path separator, such that:*

(P1)  $S = P_0 \cup P_1 \cup \dots$ , where each  $P_i$  is a set of minimum cost paths in  $G \setminus \bigcup_{j < i} P_j$ .

(P2)  $\sum_i |P_i| \leq k$ , i.e., the total number of paths in  $S$  is at most  $k$ .

(P3) either  $G \setminus S$  is empty, or each connected component of  $G \setminus S$  is  $k$ -path separable and has at most  $n/2$  vertices;

**Theorem 1 (Main)** *Every  $H$ -minor-free connected graph is  $k$ -path separable, for  $k = k(H)$ , and a  $k$ -path separator can be computed in polynomial time.*

This result has important consequences about “object location problems”. Indeed, using  $k$ -path separability property one can show:

**Theorem 2** *For every connected  $n$ -vertex weighted  $k$ -path separable graph  $G$  with aspect ratio  $\Delta$  there exists a distribution of long-range edges, computable in polynomial time, such that the greedy routing performs in  $O(k^2 \log^2 n \log^2 \Delta)$  expected number of hops.*

**Theorem 3** *For every  $n$ -vertex weighted  $k$ -path separable graph  $G$ , and for every  $\varepsilon > 0$ , there exists an  $(1 + \varepsilon)$ -approximate distance labeling scheme with  $O(k/\varepsilon \cdot \log n)$  space labels. Moreover, the labels are constructible in polynomial time, and form a distance oracle of  $O(k/\varepsilon \cdot n \log n)$  space supporting  $(1 + \varepsilon)$ -approximate distance queries in  $O(k/\varepsilon \cdot \log n)$  time.*

## 1.3 Outline of techniques

[Theorem 1](#) is the main result of this paper. Indeed, once the decomposition of the  $k$ -path separable graph is given, then many object location problems can be solved based on a generalization of the arguments of [Tho04]. For the small-world problem we use a novel potential function argument.

Let us now sketch the heart of this paper, the decomposition theorem for  $H$ -minor-free graphs stated by [Theorem 1](#). The main ingredient is the use of the tree structure of  $H$ -minor-free graphs by Robertson and Seymour [RS86, RS03].

Roughly speaking a  $H$ -minor-free graph has a tree-decomposition in “almost” embeddable subgraphs on some surfaces where  $H$  cannot be embedded. “Almost” means that each subgraph of the tree is embedded up to a constant number of vertices (called apices), and up to a constant number of disjoint non-embeddable parts (called vortices) that have bounded pathwidth and are associated with a single face. The constants depend only on  $H$ .

A general paradigm for solving problems on these graphs, for example see [DFHT05], is to first solve the problem on planar graphs, then to extend it on bounded genus, and then to extend to bounded genus graphs having vortices and apices.

There are however many technical problems due to our specific  $k$ -path property. For instance, we cannot concentrate our attention only to one subgraph of the tree-decomposition without paying attention to the whole graph, because we are looking for a set of shortest paths that potentially expand to everywhere in the (weighted) graph, even if we can effectively control the extremities of the paths. Another difficulty is to adapt known planar techniques to bounded genus graphs having vortices, remarking that the underlying surface can be orientable or nonorientable (for instance a Klein bottle), and that vortices may create nontrivial path crossings on the surface. We overcome the problems with a novel generalization of surface curves for embedded graphs with vortices: *vortex-paths*.

To identify our  $k$ -path separator in  $G$ , we proceed in three steps. In Step one, a large separating subgraph (center) of the graph is identified and some nodes (called apices) are removed. The remaining subgraph will be the place where all the shortest paths start from. In Step two, the subgraph obtained from step one, is processed in an iterative manner, each time extracting vortex-paths, reducing the Euler genus of the embedded part, until the resulting subgraph has an embedded part that is planar. To bound the number of paths, technical care is taken to use vortex-paths that intersect each vortex at most once. Step three takes the nearly planar subgraph (planar with constant number of vortices) and gives a clique-weight that captures the separation of the subgraph in the original graph. This weighting scheme generalizes the regular vertex weighting to capture the torso connectivity between bags. Using this clique-weighting, it finds vortex-paths that separate the graph into components of less than half. The two last parts of the proof are technically complicated by the requirement to deal with vortices, which are non-embeddable regions in the graph that break the traditional tools used for embeddable graphs. We believe one of our main technical contributions is our notion of *vortex-path* which plays a central role in the proof. We believe this notion has applications to other problems in  $H$ -minor-free graphs. We note that the above description is an oversimplification that hides several subtle difficulties.

The plan of this paper is as follows. Section 2 presents preliminaries to the Robertson and Seymour Theorem about the structure of  $H$ -minor-free graphs. The main proof is in Section 3 and the small world result is in Section 4. For the presentation of the main theorem some proofs of technical lemmas appear in Appendix A to Appendix E. In Section 5 we extended our result on path separators to doubling dimension isometric separators, and show some lower bounds. In particular, we show that the approach of [Tho04], the tree-separator of planar graphs, cannot be extended to  $K_6$ -minor-free graphs. Distance labels and oracles are discussed in Appendix F. We conclude in Section 6 where we leave open the object location problem for bounded degree graphs.

## 2 Preliminaries

### 2.1 The tree structure of $H$ -minor-free graphs

A *tree-decomposition* of a graph  $G$  is a tree  $\mathcal{T}$  whose vertices, called *bags*, are subsets of vertices of  $G$  such that:

1. for every vertex  $u$  of  $G$ , there exists a bag  $X$  of  $\mathcal{T}$  such that  $u \in X$ ;

2. for every edge  $\{u, v\}$  of  $G$ , there exists a bag  $X$  of  $\mathcal{T}$  such that  $u, v \in X$ ; and
3. for every vertex  $u$  of  $G$ , the set of bags containing  $u$  induces a subtree of  $\mathcal{T}$ .

An important property following from the last two points is that every path between  $u \in X$  and  $v \in Y$  in  $G$  has to intersect all the bags on the path from  $X$  and  $Y$  in  $\mathcal{T}$ . Therefore,  $X$  disconnects  $G$  if  $\mathcal{T} \setminus X$  is composed of more than one subtree.

The *width* of a tree-decomposition  $\mathcal{T}$  is  $\max_{X \in \mathcal{T}} |X| - 1$ . A *path-decomposition* of  $G$  is a tree-decomposition  $\mathcal{T}$  where  $\mathcal{T}$  is a path. The *treewidth* of  $G$  (resp. *pathwidth*) is the minimum of the width, over all tree-decompositions (resp. path-decompositions) of  $G$ .

A *joint set* in a graph having a tree-decomposition  $\mathcal{T}$  is  $X \cap Y$  for some bags  $X, Y$  of  $\mathcal{T}$ . The *torso* of  $G$  w.r.t.  $\mathcal{T}$  is the graph denoted by  $\tilde{G}$  where each joint set of  $\mathcal{T}$  is filled-in by a complete graph.

For a graph  $G$  and a subset of vertices  $X$ ,  $G[X]$  denotes the subgraph of  $G$  induced by  $X$ . We denote by  $\mathcal{T} \cap X$ , the graph obtained by intersecting each bag of  $\mathcal{T}$  with  $X$ , and by taking the subgraph of  $\mathcal{T}$  induced by the resulting nonempty bags. If  $G[X]$  is connected, then  $\mathcal{T} \cap X$  is a tree-decomposition of  $G[X]$ .

A graph  $G$  is a *vortex* if there is a sequence of distinct vertices  $u_1, \dots, u_t$  of  $G$ , called *perimeter*, and a path-decomposition of  $G$  whose bags are  $X_1, \dots, X_t$  ordered such that the edges of the path are the pairs  $\{X_i, X_{i+1}\}$  and  $u_i \in X_i$  for all  $i$ . The width of a vortex is the width of its path-decomposition.

A *surface* is a non-null compact connected 2-manifold without boundary. A face of a graph embedded on a surface is *cellular* if it is homeomorphic to an open disc.

A graph  $G$  is *h-almost embeddable* on a surface  $\Sigma$  if there exists a set  $X$  of at most  $h$  vertices, called *apices*, such that  $G \setminus X = G_\Sigma \cup W_1 \cup \dots \cup W_t$ ,  $t \leq h$  such that:

1. the graph  $G_\Sigma$  has an embedding on  $\Sigma$ ;
2. the graphs  $W_i$ 's are pairwise disjoint vortices of width at most  $h$ ;
3. the perimeter of each vortex is the border of some cellular face of  $G_\Sigma$ .

An *h-nearly planar* graph is a graph  $h$ -almost embeddable on the sphere with no apices.

It is known that every graph excluding a planar minor  $H$  has bounded treewidth [RS86]. The best known upper bound on the treewidth is  $20^{2(2|V(H)+4|E(H)|)^5}$  [RST94], which potentially is greater than  $20^{10^6 r^5}$  for a maximal planar graph  $H$  on  $r$  vertices. Actually, a much more general result, due to [RS03], gives us the structure of  $H$ -minor-free graphs for any graph  $H$ , that can be expressed as follows (see also [DT99, Gro03]):

**Theorem 4 (Robertson & Seymour [RS86, RS03])** *For every  $H$ -minor-free graph  $G$ , there is a number  $h = h(H)$  and a tree-decomposition  $\mathcal{T}$  such that for every bag  $X$  of  $\mathcal{T}$ , either  $|X| \leq h$ , or  $\tilde{G}[X]$  is  $h$ -almost embeddable on a surface on which  $H$  is not embeddable.*

A tree-decomposition  $\mathcal{T}$  satisfying Theorem 4 can be constructed in polynomial time for fixed  $H$ , where the exponent depends on the number  $h$ . The algorithm comes directly from [RS95] (cf. [Gro03, Lemma 15]). More recently, another algorithm has been presented in [DFHT05]. Note that the so constructed tree-decompositions have size linear in  $G$ , and the faces of all the embeddings are cellular.

### 3 Finding a $k$ -path separator

In this paragraph we assume that  $G$  is a weighted connected  $H$ -minor-free graph with  $n$  vertices, and that  $\mathcal{T}$  is a linear size tree-decomposition for  $G$  satisfying [Theorem 4](#). Let  $h$  be the constant, depending on the fixed graph  $H$ , involved in [Theorem 4](#).

The family of graphs excluding a fixed minor is closed under minor taking, and thus under induced subgraph. Therefore, to prove [Theorem 1](#) it suffices to prove that  $G$  has a  $k$ -path separator  $S$  without proving that the small ( $\leq n/2$  vertices) connected components of  $G \setminus S$  are  $k$ -path separable.

To find such a separator  $S = P_0 \cup P_1 \cup \dots$ , we proceed iteratively in several steps. At a given step  $s$ , we compute a set of minimum cost paths  $P_s$  in the current graph  $G_s$ , and we keep the largest connected component  $G_{s+1}$  of  $G_s \setminus P_s$ , until all of them have  $\leq n/2$  vertices.

The process is split into three main steps detailed below. Initially, we set  $s = 0$ , and  $G_0 = G$ ,  $\mathcal{T}_0 = \mathcal{T}$ .

#### Step 1: Remove the center apices

The next classical result holds for every tree-decomposition of any graph  $G$  and any tree-decomposition  $\mathcal{T}$  of  $G$ .

**Lemma 5** *There is a bag  $C$  of  $\mathcal{T}$ , call center, such that the connected components of  $G \setminus C$  have at most  $n/2$  vertices.*

Let  $C_0$  be the bag given by [Lemma 5](#) applied on  $G_0$  and  $\mathcal{T}_0$ .  $C_0$  can be found in polynomial time, since  $\mathcal{T}_0$  has linear size. If  $|C_0| \leq h$ , then we set  $S = P_0 = C_0$ , and we have proved that  $S$  is a  $h$ -path separator for  $G_0$ . Otherwise, let  $P_0$  be the set of apices of  $C_0$ , and let  $G_1$  be a largest connected component of  $G_0 \setminus P_0$ . Since only  $G_1$  can have more than  $n/2$  vertices, it suffices to concentrate our attention on  $G_1$ , and to complete  $P_0$  with some sequence of minimal cost paths  $P_1, P_2, \dots$  to form the wanted separator  $S$  for  $G_0$ .

Let  $\mathcal{T}_1 = \mathcal{T}_0 \cap G_1$ , and  $C_1 = G_1 \cap C_0$ .  $\mathcal{T}_1$  is a tree-decomposition for  $G_1$ , and  $C_1$  is a bag of  $\mathcal{T}_1$  such that its torso,  $\tilde{C}_1$ , is  $h$ -almost embeddable with no apices.

#### Step 2: Make the center nearly planar

The goal of this step is to transform  $\tilde{C}_1$  into a nearly planar graph by removing iteratively some set of minimum cost paths. For that we need some terminologies and definitions.

Let  $\Sigma_0$  be the surface on which  $\tilde{C}_0$  is  $h$ -almost embedded, denote by  $C_{\Sigma_0} = \tilde{C}_0 \cap \Sigma_0$  the part of  $\tilde{C}_0$  embedded on  $\Sigma_0$ , and let  $g$  be the *Euler genus* of  $\Sigma$  we define hereafter. The Surface Classification Theorem states that every surface is homeomorphic to a space obtained from the sphere by adding handles or crosscaps. The *Euler genus* of  $\Sigma_0$  is either  $\lambda$  if  $\Sigma_0$  was obtained by adding  $\lambda \geq 0$  handles (orientable surface), or  $2\mu$  if  $\Sigma_0$  was obtained by adding  $\mu \geq 1$  crosscaps (nonorientable surface). For an introduction to terminologies about graphs on surfaces we refer to [\[MT01\]](#).

Note that in general there is no upper bound<sup>2</sup> of the Euler genus of  $\Sigma_0$  in terms of  $h$ . However,

<sup>2</sup>E.g., consider  $r \geq 7$  such that  $r(r-7) \equiv 0 \pmod{12}$  ( $r = 7, 12, 15, 16, 19, \dots$ ). Then, any graph having  $K_r$  as its

an important observation is that  $g < |E(H)|$ , because every graph  $X$  with a cellular embedding on  $\Sigma$  satisfies  $g \leq |E(X)| - |V(X)| + 1$  from Euler's Formula.

A graph is  $(h, p, g)$ -embeddable if it is  $h$ -almost embeddable with no apices on a surface  $\Sigma$  of Euler genus at most  $g$ , and if the width of its vortices is at most  $p \leq h$ . So  $\tilde{C}_1$  is  $(h, h, g)$ -embeddable, and  $h$ -nearly planar graphs are exactly the  $(h, h, 0)$ -embeddable graphs. In the following, we assume  $G$  is a  $(h, p, g)$ -embeddable graph, and let  $G_\Sigma$  denote its embedded part.

A *vortex-path* is a subgraph  $\mathcal{V}$  of  $G$  that can be decomposed in  $\mathcal{V} = Q_0 \cup X_1 \cup Y_1 \cup Q_1 \cup \dots \cup X_t \cup Y_t \cup Q_t$  such that for every  $i \in \{0, \dots, t\}$ :

1.  $Q_i$ , called *segment* of  $\mathcal{V}$ , is a path wholly contained in  $G_\Sigma$ ;
2. there are pairwise disjoint vortices  $W_j$ 's of  $G$  such that  $X_i$  and  $Y_i$  are bags of  $W_i$ ;
3. One extremity of  $Q_i$  is the perimeter vertex of  $Y_i$  (for  $i > 0$ ) and the other one is the perimeter vertex of  $X_{i+1}$  (for  $i < t$ ), and no other vertices of  $Q_i$  is a perimeter vertex.

The *projection* of a vortex-path  $\mathcal{V}$ , denoted by  $\bar{\mathcal{V}}$ , is the path formed by  $Q_0 \cup e_1 \cup Q_1 \cup \dots \cup e_t \cup Q_t$  where  $e_i, i > 0$ , is an extra edge added to  $G$  between the perimeter vertex of  $X_i$  and the perimeter vertex of  $Y_i$ , and embedded on the face (which is cellular) of the vortex  $W_i$ . We observe that the projection of  $\mathcal{V}$  is a curve of  $\Sigma$  (cf. Fig. 1(b)).

With every path  $P$  of  $G$  whose extremities are in  $G_\Sigma$ , one can associate a vortex-path  $Q_0 \cup X_1 \cup Y_1 \cup \dots \cup Q_t$  defined as follows (see Fig. 1): Start a walk on  $P$  from one extremity until encountering the first perimeter vertex, say  $x_1$ . This first part of  $P$  forms the segment  $Q_0$ , and the vortex bag corresponding to  $x_1$  is  $X_1$ . Then, continue the walk on  $P$  and select  $y_1$  to be the last perimeter vertex on  $P$  belonging to the same vortex that  $x_1$  belongs to. This forms  $Y_1$ , the bag whose perimeter vertex is  $y_1$ . Then, continue along  $P$  up to the next perimeter vertex, forming segment  $Q_1$ , and so on. Observe that the part of  $P$  between  $x_1$  and  $y_1$  may enter and leave many vortices, each one several times. However, by construction, the vortex-path of  $P$  enters and leaves pairwise distinct vortices.

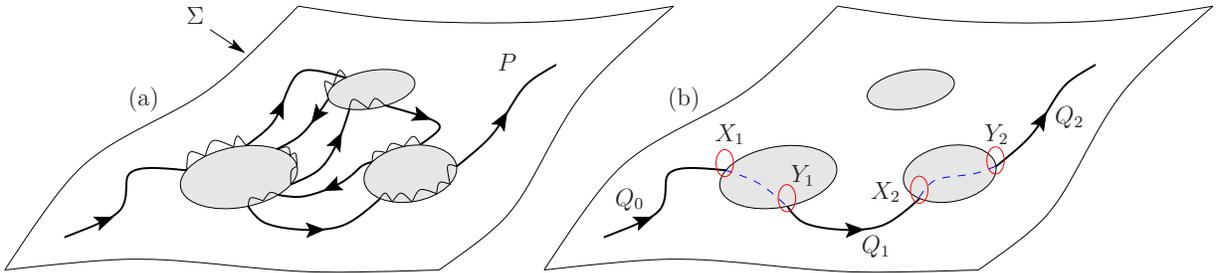


Figure 1: (a) A path  $P$  intersecting three vortices; (b) its vortex-path  $\mathcal{V} = Q_0 \cup X_1 \cup Y_1 \cup \dots \cup Q_2$ , and its projection.

The path from  $u$  to  $v$  in a rooted tree  $T$  is denoted by  $T(u, v)$ , and  $T(u, v)$  is said to be *monotone* if  $u$  and  $v$  are relatives, i.e.,  $u$  is ancestor of  $v$  or  $v$  is ancestor of  $u$ .

We have the following result which is the key of Step 2:

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bi-components, excludes  $K_{r+1}$  and has a tree-decomposition of 0-almost embeddable graphs on a orientable surface of genus  $1 + r(r - 7)/12 = \Omega(r^2)$  (this value is indeed the genus of  $K_r$ ).

**Lemma 6** *Let  $G$  be a  $(h, p, g)$ -embedded graph for  $g > 0$ , and let  $T$  be a spanning tree of  $G$  rooted in the embedded part of  $G$ . Then, there exist two vortex-paths  $\mathcal{V}_1, \mathcal{V}_2$  such that each segment is a monotone path of  $T$ , and each connected component of  $G \setminus (\mathcal{V}_1 \cup \mathcal{V}_2)$  is  $(h + 1, p, g - 1)$ -embedded. Moreover,  $\mathcal{V}_1, \mathcal{V}_2$  and the embeddings can be computed in polynomial time.*

We will now carefully apply iteratively [Lemma 6](#) to the graph  $\tilde{C}_1$  in order to get a  $(h + g)$ -nearly planar graph. For that we assume the following:

1.  $G_s$  is connected;
2.  $\mathcal{T}_s$  is tree-decomposition of  $G_s$ ;
3.  $C_s$  is a bag of  $\mathcal{T}_s$ ; and
4.  $\tilde{C}_s$  is  $(h - 1 + s, h, g + 1 - s)$ -embedded (the torso is w.r.t.  $\mathcal{T}_s$ ).

For  $s = 1$  we check all the conditions hold. Here is the main loop:

**Main loop:**

1. If  $C_s \cap C_{\Sigma_0} = \emptyset$  or  $\tilde{C}_s$  is  $(h - 1 + s, h, 0)$ -embedded, we jump directly to Step 3. The former case may occur if, for instance,  $C_s$  is a subgraph of  $C_0$ 's vortices.
2. Choose a root  $r_s \in C_s \cap C_{\Sigma_0}$ , and perform from  $r_s$  a minimum cost path tree  $A_s$  in  $G_s$  spanning the vertices of  $C_s$ .
3. Let  $T_s$  be the graph  $A_s \cap C_s$  augmented with the edges between any two vertices  $u, v$  satisfying the two conditions:
  - (a)  $u$  and  $v$  are both in the same joint set of  $C_s$ ; and
  - (b)  $A_s(u, v)$  is monotone and  $A_s(u, v) \cap C_s = \{u, v\}$ .

**Lemma 7**  $T_s$  is a spanning tree of  $\tilde{C}_s$ .

Note that by construction, for any pair  $u, v \in C_s$ ,  $V(T_s(u, v)) \subseteq V(A_s(u, v))$ , and that if  $T_s(u, v)$  is monotone, then so is  $A_s(u, v)$ .

4. We now apply [Lemma 6](#) to the graph  $\tilde{C}_s$  with the spanning tree  $T_s$ , and we obtain the vortex-paths  $\mathcal{V}_j = Q_0^j \cup X_1^j \cup Y_1^j \cup \dots \cup Q_{t_j}^j$  for  $j \in \{1, 2\}$ . Assume that each segment  $Q_i^j$  is a path going from  $u_i^j$  to  $v_i^j$ . By [Lemma 6](#),  $Q_i^j = T_s(u_i^j, v_i^j)$  is monotone.
5. Finally, we update  $S = S \cup P_s$  where:

$$P_s = \bigcup_{j \in \{1, 2\}} \left( A_s(u_0^j, v_0^j) \cup \bigcup_{i=1}^{t_j} X_i^j \cup Y_i^j \cup A_s(u_i^j, v_i^j) \right).$$

6. It is clear that  $\mathcal{V}_1 \cup \mathcal{V}_2 \subseteq P_s$ , since  $A_s(u_i^j, v_i^j)$  includes the segment  $T_s(u_i^j, v_i^j) = Q_i^j$  for all  $i, j$ . Therefore, according [Lemma 6](#), each connected components of  $G_s \setminus P_s$  is  $(h - 1 + s + 1, h, g + 1 - s - 1)$ -embedded, i.e.,  $(h - 1 + (s + 1), h, g + 1 - (s + 1))$ -embedded.

7. Observe that  $P_s$  is a set of minimum cost paths in  $G_s$ . Indeed,  $A_s(u_i^j, v_i^j)$  is monotone in  $A_s$  (by monotonicity of  $T_s(u_i^j, v_i^j)$ ), and thus is a minimum cost path. The number of paths in  $P_s$  is at most  $2 + (t_1 + t_2) \cdot (2p + 1)$  because each vortex bag has at most  $p + 1$  vertices (we count only  $p$  vertices since the perimeter vertex is a part of an entering or leaving segment). In a vortex-path, the number of segments cannot exceed the total number of vortices, so by assumption  $t_j \leq h - 1 + s$ . We have also that  $p \leq h$ , therefore  $|P_s| \leq 2 + 2(h - 1 + s) \cdot (2h + 1)$ .
8. Let  $G_{s+1}$  be the largest connected component of  $G_s \setminus P_s$ . Let  $\mathcal{T}_{s+1} = \mathcal{T}_s \cap G_{s+1}$ , and  $C_{s+1} = G_{s+1} \cap C_s$ .  $\mathcal{T}_{s+1}$  is a tree-decomposition for  $G_{s+1}$ , and  $C_{s+1}$  is a bag of  $\mathcal{T}_{s+1}$ . An important observation is that the joint sets of  $\tilde{C}_{s+1}$  are included in those of  $\tilde{C}_s$  since cliques of  $\tilde{C}_s$  were not split by the removal of  $P_s$ . Therefore  $\tilde{C}_{s+1}$  corresponds to a component of the graph computed from  $\tilde{C}_s$  by [Lemma 6](#). Therefore,  $\tilde{C}_{s+1}$  is  $(h - 1 + (s + 1), h, g + 1 - (s + 1))$ -embedded as required.

All the pre-conditions of the main loop are satisfied, and we continue with  $s = s + 1$ .

### Step 3: Split the nearly planar center

At this stage  $s \leq g + 1$ , and  $S = P_0 \cup \dots \cup P_{s-1}$ . Currently the total number of minimum cost paths in  $S$  is bounded by

$$h + \sum_{i=1}^{s-1} |P_i| \leq h + \sum_{s=1}^g (2 + 2(h - 1 + s) \cdot (2h + 1)) = O(hg(h + g)) .$$

Moreover, either  $C_s \cap C_{\Sigma_0} = \emptyset$ , or  $\tilde{C}_s$  is  $(h + g, h, 0)$ -embeddable, i.e.,  $\tilde{C}_s$  is  $(h + g)$ -nearly planar. We eliminate the former case.

**Lemma 8** *If  $C_s \cap C_{\Sigma_0} = \emptyset$ , then there was a vortex bag  $X$  of  $C_0$  such that the connected components of  $G_s \setminus X$  have at most  $n/2$  vertices.*

According to [Lemma 8](#), if  $C_s \cap C_{\Sigma_0} = \emptyset$ , we can find in polynomial time the vortex bag  $X$ , and set  $P_s = X \cap G_s$ . As  $X \cap G_s$  is a proper subset of the vortex bag  $X$  — indeed the perimeter vertex of  $X$  was belonging to  $C_{\Sigma_0}$  and thus is missing from  $C_s$  — we get  $|P_s| = |X \cap G_s| \leq h$ . Thus we have proved that  $S = P_0 \cup \dots \cup P_s$  is a  $k$ -path separator for  $G$  for  $k = O(hg(h + g))$  which is bounded by some function depending only on  $H$  since we have seen that  $g < |E(H)|$ .

Our goal now is to find a separator in  $C_s$  that also splits  $G_s$  in small components. For that we need some definitions and the use of two lemmas.

A *clique-weight* for a graph  $G$  is a pair  $(\mathcal{K}, \omega)$  where  $\mathcal{K}$  is a set of cliques of  $G$ , and  $w$  a function associating with each clique  $K \in \mathcal{K}$  a real  $\omega(K) \geq 0$ . For every subgraph  $A$  of  $G$  having a clique-weight  $(\mathcal{K}, \omega)$ , we define the *weight* of  $A$  as:

$$f(A) = \sum_{\substack{K \cap A \neq \emptyset \\ K \in \mathcal{K}}} \omega(K) .$$

A clique-weight for  $G$  is a non-trivial generalization of a vertex-weight function. This can be seen by setting  $\mathcal{K}$  as the collection of cliques each one composed of one single vertex of  $G$ . So

that  $\omega(\{u\})$  is the weight of vertex  $u$ , and  $f(A)$  equals the sum over the vertices of  $A$  of the function  $\omega$ . Unlike vertex-weight functions, observe that if  $A \cup B \subseteq G$  and  $A \cap B \neq \emptyset$ , then  $f(A) + f(B) \not\leq f(G)$  for general clique-weights (for instance if there is a clique intersecting  $A$  and  $B$ ). However,  $f(A) + f(B) \leq f(G)$  whenever  $A$  and  $B$  are connected components of  $G \setminus S$  for any  $S$ .

A *half-size separator* for  $G$  with clique-weight  $(\mathcal{K}, \omega)$  is a set of vertices whose removal leaves connected components of weight at most half the weight of  $G$ , i.e., of weight at most  $f(G)/2$ .

The motivation for this generalization is the following simple result, that states that every half-separator, for a suitable clique-weight, is actually the wanted separator for the whole graph.

**Lemma 9** *Let  $C$  be a center of tree-decomposition of an  $n$ -vertex graph  $G$ . Then, in polynomial time, one can construct a clique-weight  $(\mathcal{K}, \omega)$  for the torso  $\tilde{C}$  such that, given any half-size separator  $S$  of  $\tilde{C}$ , the connected components of  $G \setminus S$  have at most  $n/2$  vertices.*

So virtually, we can forget for a while about the small connected components joining  $C$  and concentrate our attention on half-size separators of  $\tilde{C}$ .

The following result is non-trivial variant of the Lipton-Tarjan planar separator [LT79] which is a generalization of the three-leaves tree separator of Thorup [Tho04], applied to a more general graph and with a more general vertex-weight function. The latter result of [Tho04] is a corollary of Lemma 10 if  $G$  is planar, i.e., if  $G$  is 0-nearly planar.

**Lemma 10** *Let  $G$  be a nearly planar graph with a clique-weight and a spanning tree  $T$  rooted in the planar part of  $G$ . In polynomial time, one can construct a half-size separator for  $G$  composed of at most three vortex-paths whose segments consist of monotone paths of  $T$ .*

### Final construction:

At this step  $C_s \cap C_{\Sigma_0} \neq \emptyset$ . So, we went out from Step 2 because  $\tilde{C}_s$  is  $(h+g)$ -nearly planar. Let  $A_s$  and  $T_s$  be the trees constructed as in Point 2 and Point 3 of the main loop of Step 2.  $A_s$  is a minimum cost tree rooted at some vertex of the planar part of  $G_s$ , and by Lemma 7,  $T_s$  is a spanning tree of  $\tilde{C}_s$ .

We now construct the clique-weight  $(\mathcal{K}, \omega)$  for  $\tilde{C}_s$  in  $G_s$  (with tree-decomposition  $\mathcal{T}_s$ ) as done in Lemma 9, and we apply Lemma 10 to the  $(h+g)$ -nearly planar graph  $\tilde{C}_s$  and its spanning tree  $T_s$ .

Let  $\mathcal{V}_1, \mathcal{V}_2, \mathcal{V}_3$  be the vortex-paths constructed in Lemma 10, where  $\mathcal{V}_j = Q_0^j \cup X_1^j \cup Y_1^j \cup \dots \cup Q_{t_j}^j$  with  $Q_i^j = T_s(u_i^j, v_i^j)$  for all  $i, j$ . Similarly to Point 5 of Step 2, we update the separator  $S = S \cup P_s$  with:

$$P_s = \bigcup_{j \in \{1, 2, 3\}} \left( A_s(u_0^j, v_0^j) \cup \bigcup_{i=1}^{t_j} X_i^j \cup Y_i^j \cup A_s(u_i^j, v_i^j) \right).$$

Note that  $\mathcal{V}_1 \cup \mathcal{V}_2 \cup \mathcal{V}_3 \subseteq P_s$ , since  $A_s(u_i^j, v_i^j)$  includes the segment  $T_s(u_i^j, v_i^j) = Q_i^j$  for all  $i, j$ . Therefore, according to Lemma 10,  $P_s$  is a half-size separator of  $\tilde{C}_s$ . By Lemma 9, all the connected components of  $G_s \setminus P_s$  have at most  $|V(G_s)|/2 \leq n/2$  vertices.

We complete the proof by observing that  $P_s$  is a set of minimum cost paths in  $G_s$ , as  $A_s(u_i^j, v_i^j)$  is monotone and thus is a minimum cost path in  $G_s$  for all  $i, j$ . The number of paths in  $P_s$  is at most  $3 + (t_1 + t_2 + t_3) \cdot (2(h+g) + 1) \leq 3 + 3(h+g) \cdot (2(h+g) + 1) = O(hg(h+g))$ .

In all the cases we have proved that  $S = P_0 \cup \dots \cup P_s$  is a  $k$ -path separator for  $G_0$  for  $k = O(hg(h+g)) = O(h|E(H)| \cdot (h + |E(H)|))$ , which is bounded by some function of  $H$  as claimed. This completes the proof of [Theorem 1](#).

**Note.** The above proof of [Theorem 1](#) can be strengthened to construct a  $k$ -path vertex-weighted separator, that is a separator  $S$  that splits  $G$  (having edge and vertex-weights) in components of vertex-weight at most half of the total vertex-weight of  $G$ ,  $S$  still composed of minimum cost paths as defined by property (P1). For that, lemmas [5](#) and [9](#) can be easily adapted.

## 4 Small-World, Proof of [Theorem 2](#)

Given a weighted graph  $G$  and  $u, v \in V(G)$ , let  $d_G(u, v)$  be the cost of a minimal cost path in  $G$  between  $u$  and  $v$ . We omit the subscript  $G$  when it is clear from the context. Distance is extended naturally to sets, given sets  $U, V \subseteq V(G)$ , let  $d(U, V) = \min\{d(u, v) \mid u \in U, v \in V\}$ .

Let  $G$  be a weighted  $n$  node  $k$ -path separable graph with aspect ratio  $\Delta = \max d(u, v) / \min d(u, v)$ . To avoid dragging a normalization constant we assume  $\min d(u, v) = 1$ . For any subgraph  $H$  of  $G$ , let  $S(H)$  be the set of sets of paths that separate  $H$  in to small components (i.e., of size at most  $|V(H)|/2$ ).

We build a rooted tree  $\mathcal{T}$  called the *decomposition tree* of  $G$  as follows. The nodes of  $\mathcal{T}$  are subgraphs of  $G$ . The root node of  $\mathcal{T}$  is  $G$ . For any  $H \in V(\mathcal{T})$ , its children  $J_1, \dots, J_t$  are the connected components induced by  $H \setminus S(H)$ . Observe that the depth of  $\mathcal{T}$  is at most  $\log n$ .

**The distribution.** For a given node  $v \in G$ , let  $H_1(v), H_2(v), \dots, H_r(v)$  be the path in  $\mathcal{T}$  starting from the root  $G$  containing all the nodes  $H \in V(\mathcal{T})$  such that  $v \in H$ . Node  $v$  chooses uniformly at random  $\tau \in \{1, \dots, r\}$ .

Given  $\tau$  the node chooses uniformly at random one path  $Q$  out of the paths in  $S(H_\tau(v))$ . Given the path  $Q \in P_i \in S(H_\tau(v))$  and the graph  $J = H \setminus \bigcup_{j < i} P_j$  we define the following landmarks  $L = L(Q) \subseteq Q$ .

Let  $Q = x_1, \dots, x_t$  be the nodes on the path. Let  $x_c \in Q$  be a node such that  $d_J(v, x_c) = d_J(v, Q)$ . Let  $d = d_J(v, x_c)$ . Now we consider separately the paths  $Q_{(-1)} = x_1, \dots, x_c$  and  $Q_{(+1)} = x_c, \dots, x_t$ . For each  $j \in \{-1, 1\}$ : (1) For each  $i \in \{0, 1, 2, 3, \dots, 10\}$  add to  $L$  the first node in  $Q$  such that  $d_{Q_{(j)}}(x_c, x) \geq (i/2) \cdot d$ . (2) For each  $i \in \{0, 1, 2, 3, \dots, \lceil \log \Delta \rceil\}$  add to  $L$  the first node in  $Q$  such that  $d_{Q_{(j)}}(x_c, x) \geq 2^i \cdot d$ .

Observe that  $|L| = O(\min\{t, \log \Delta\})$ . After node  $v$  chooses a component index  $\tau$  and chooses a path  $Q$ , it chooses uniformly at random a landmark  $\ell$  in  $L = L(Q)$ . This completes the distribution for node  $v$ .

**Claim 1** *Let  $L$  be the set of landmarks chosen by node  $v$  in graph  $J$  in respect to path  $Q$ .*

*For any  $x \in Q$ , there exists  $\ell \in L$  such that  $d_Q(\ell, x) \leq \frac{3}{4}d_J(v, x)$ .*

**Proof.** Let  $d = d_J(v, x_c)$ . There are two cases to consider. If  $d_J(v, x) \leq 4d$  then  $d_Q(x_c, x) \leq 5d$  since  $Q$  is a shortest path in  $J$ . Therefore there exists  $0 \leq i \leq 10$  such that  $\ell \in L$  is the first node on  $Q$  with  $d_Q(x_c, \ell) > id/2$  and  $d_Q(\ell, x) \leq d/2 \leq d_J(v, x)/2$  since  $d_J(v, x) \geq d$ .

Otherwise if  $d_J(v, x) > 4d$  then  $\frac{3}{4}d_J(v, x) \leq d_J(v, x) - d \leq d_Q(x_c, x) \leq d_J(v, x) + d \leq \frac{5}{4}d_J(v, x)$ . Let  $i \in \{0, 1, 2, 3, \dots, \lceil \log \Delta \rceil\}$  be the index such that  $d_J(v, x)/2 < 2^i d \leq d_J(v, x)$  then for the corresponding  $\ell \in L$  with  $d_Q(x_c, \ell) \geq 2^i d$  we have  $d_Q(\ell, x) \leq \frac{3}{4}d_J(v, x)$ .  $\square$

## Augmenting a graph with a distribution bundle

**Definition 2 (Distribution bundle)** A distribution bundle is a function  $\mathcal{D} : V \times V \rightarrow \mathbb{R}^+$  such that for any  $v \in V$  the function  $\mathcal{D}(v, \cdot)$  is a distribution function on  $V$  ( $\forall v \in V, \sum_{u \in V} \mathcal{D}(v, u) = 1$ ).

**Definition 3** Given a weighted undirected graph  $G = (V, E, \omega)$  and a distribution bundle  $\mathcal{D}$ , let  $\langle G, \mathcal{D} \rangle$  be a distribution on graphs with node set  $V$  formed by adding to  $E$ , for each node  $v \in V$  one directed edge  $e = (v, u)$  where  $u$  is independently chosen via  $\mathcal{D}(v, \cdot)$  and setting the weight of the edge to be  $\omega(e) = d_G(v, u)$ .

**Theorem 11** For any  $n$ -node  $k$ -path-sparable weighted graph  $G$  with aspect ratio  $\Delta$ , the expected hop count of greedy routing on  $\langle G, \mathcal{D} \rangle$  is  $O(k^2 \log^2 n \log^2 \Delta)$ , for  $\mathcal{D}$  described as above.

**Proof.** Consider any target  $t \in V(G)$ . Let  $\mathcal{T}$  be the decomposition tree of  $G$ . Let  $H_1, H_2, \dots, H_r$  be the path in  $\mathcal{T}$  starting from the root  $G$  containing all the nodes  $H \in V(\mathcal{T})$  such that  $t \in H$ . For each  $H_i$ , let  $Q_1^i, \dots, Q_k^i$  be the paths (possibly empty or one node paths) of  $S(H_i)$ . Let  $\mathcal{P} = \left\{ Q_j^i \mid 1 \leq i \leq r, 1 \leq j \leq k \right\}$ , observe that  $|\mathcal{P}| = O(k \log n)$  and  $\mathcal{P} \neq \emptyset$ .

For any  $Q \in \mathcal{P}$ , let  $x(Q) \in Q$  be a node such that  $d(x(Q), t) = \min_{y \in Q} d(y, t)$ . Let  $\Phi : V \times \mathcal{P} \rightarrow \mathbb{R}$  be the function:

$$\Phi(u, Q) = \max\{d(u, t) - d(x(Q), t), 0\}.$$

The following is immediate from the definition of  $\Phi$ .

### Claim 2

- (1) If  $d(v, t) \leq d(u, t)$  then  $\Phi(v, Q) \leq \Phi(u, Q)$  for all  $Q \in \mathcal{P}$ .
- (2) If  $\forall Q \in \mathcal{P} : \Phi(v, Q) = 0$  then  $v = t$ .

Consider a current node  $u$  that wants to reach  $t \neq u$ . Let  $R$  be a shortest path on  $G$  from  $u$  to  $t$ . Let  $Q \in \mathcal{P}$  be the first path that intersects  $R$ , formally  $Q = \operatorname{argmin}_{Q \in \mathcal{P}} \{d_R(u, Q)\}$ . If  $\Phi(u, Q) = 0$ , then let  $Q$  be the next path in  $\mathcal{P}$  that intersects  $R$ . Observe that  $\Phi(u, Q) > 0$ . We will now show that with probability  $O(k \log n \log \Delta)^{-1}$  we reach a node  $v$  such that  $\Phi(v, Q) \leq \frac{3}{4}\Phi(u, Q)$ .

Let  $i, \tau$  be the indexes such that  $Q \in P_i \in S(H_\tau(u))$ . Let  $J = H_\tau(u) \setminus \bigcup_{j < i} P_j$ . Since  $Q$  is the first path to intersect  $R$  then  $d(u, x(Q)) = d_J(u, x(Q))$ .

Hence with probability  $O(k \log n \log \Delta)^{-1}$  node  $u$  chooses  $\tau \in O(\log n)$  and  $Q \in O(k)$  and the landmark  $\ell \in O(\log \Delta)$  from [Claim 1](#) that covers  $x(Q)$ . If this happens then  $u$  has an edge to some node  $\ell$  such that  $d(\ell, x(Q)) \leq d_Q(\ell, x(Q)) \leq \frac{3}{4}d_J(u, x(Q)) = \frac{3}{4}d(u, x(Q))$ .

Hence, given such an  $\ell$ , any greedy step that  $u$  performs will lead to a node  $v$  such that  $\Phi(v, Q) \leq \Phi(\ell, Q) \leq \frac{3}{4}\Phi(u, Q)$ .

Since such an event can happen at most  $O(\log \Delta)$  times for each of the  $O(k \log n)$  paths in  $\mathcal{P}$  then by linearity the expected greedy diameter is at most  $O(k^2 \log^2 n \log^2 \Delta)$ .  $\square$

**Note 1:** when all separator paths are actually one node (as in the case of bounded treewidth graphs) then the greedy diameter can be reduced to  $O(k^2 \log^2 n)$  by setting  $L(Q)$  to simply be the single node of  $Q = \{q\}$ .

**Note 2:** if all separator paths form a graph with diameter  $k$  and additionally the graph is unweighted (as in the case of unweighted bounded treewidth graphs) then the greedy diameter reduces to  $O(\log^2 n + k \log n)$ . After node  $u$  chooses  $\tau$ , instead of randomly choosing one of the  $k$  node in  $S(H_\tau(u))$  it chooses the closest node from  $S(H_\tau(u))$ . Now consider a target  $t$  and a current node  $u$  and let  $H$  be the minimal component of  $\mathcal{T}$  that contains both. If  $d_H(u, S(H)) \leq 2k$  then  $u$  will cross  $S(H)$  after  $O(k)$  steps. Otherwise, with probability  $O(\log n)^{-1}$ , node  $u$  will have a long range contact to  $x$  the closest node in  $S(H)$ . Since  $d_H(u, S(H)) > 2k$  then  $d_H(x, t) \leq k + d_H(S(H), t)$ . Hence if this event occurs then this level of the hierarchy will be crossed in  $O(k)$  steps. Hence the expected number of hops to cross a separator is  $O(\log n + k)$ .

**Note 3:** if  $G$  is not  $k$ -path separable but is  $(k, \alpha)$ -doubling separable (see Section 5) then instead of choosing  $O(\log \Delta)$  landmarks we choose  $O(2^{O(\alpha)} \log \Delta)$  landmarks using the construction of Slivkins [Sli05]. The expected greedy diameter becomes  $O(2^{O(\alpha)} k^2 \log^2 n \log^2 \Delta)$ .

**Corollary 12** *For any  $n$ -node graph  $G$  there exists a long range link augmentation  $\mathcal{D}$  such that the expected hop count of greedy routing on  $\langle G, \mathcal{D} \rangle$  is*

1.  $O(k^2 \log^2 n)$  for weighted bounded treewidth graphs.
2.  $O(k \log n + \log^2 n)$  for unweighted bounded treewidth graphs.
3.  $O(2^{O(\alpha)} k^2 \log^2 n \log^2 \Delta)$  for  $(k, \alpha)$ -doubling separable graphs with aspect ratio  $\Delta$ .

## 5 About $k$ -Path Separability of Graphs

### 5.1 Lower Bounds for Sparse Graphs

Although our results applies to a large family of sparse graphs, graphs excluding  $K_r$  have  $O(r\sqrt{\log r} \cdot n)$  edges [Tho01], no techniques can achieve similar performances for general sparse graphs. More precisely:

**Theorem 13** *For every fixed  $0 \leq \varepsilon < 2$ , there are unweighted graphs with at most  $n$  vertices and with  $O(n)$  edges on which every stretch- $(1 + \varepsilon)$  distance labeling scheme with  $L$ -bit labels, or on which every stretch- $(1 + \varepsilon)$  labeled routing scheme with  $L$ -bit labels and routing tables implies  $L = \Omega(\sqrt{n})$ .*

*In particular such sparse graphs are not  $k$ -path separable for  $k = o(\sqrt{n}/\log^2 n)$ .*

**Proof.** Plug the classical lower bounds into a graph of  $N = \lceil \sqrt{n} \rceil$  vertices and so with less than  $N^2 = O(n)$  edges. It is known that, for every stretch  $s < 3$ , there are unweighted  $N$ -vertex graphs for which every stretch- $s$  routing scheme requires  $\Omega(N)$ -bit labels [GG01], or for which every stretch- $s$  distance labeling scheme requires  $\Omega(N)$ -bit labels [GP04, TZ05].

Let us fix  $\varepsilon = 1$  (or any value  $0 < \varepsilon < 2$ ). From Theorem 3, every  $k$ -separable graph supports a 2-approximate distance labeling scheme with  $L$ -bit labels where  $L = O(k \log^2 n)$ . So for the above sparse graph,  $L = \Omega(\sqrt{n})$ , and thus it cannot be  $k$ -separable for  $k = o(\sqrt{n}/\log^2 n)$ .  $\square$

## 5.2 Strongly separable graphs

Let us first observe that the graph composed of a path of  $n/2$  vertices and of a stable of  $n/2$  vertices with all edges between the path and stable vertices, contains  $K_{n/2, n/2}$  as minor and is a 1-path separable (the whole path) if the path edges have weight 1 and the other edges weight  $n/2$ . Thus the family of  $O(1)$ -path separable graphs does not reduce to  $K_{O(1)}$ -minor-free graphs.

A  $k$ -path separator  $S$  for  $G$  is said *strong* if  $S = P_1$ , i.e.,  $S$  reduces to the union of  $k$  minimum cost paths in  $G$ . Although [Tho04] proved that planar graphs are strongly 3-path separable, we stress that for a fixed  $k$  this natural definition actually captures much less graphs. For instance, we construct some  $K_6$ -minor-free graphs that are not strongly  $\sqrt{n}$ -path separable, in contrast Theorem 1 implies that they are  $O(1)$ -path separable.

Unless explicitly expressed, graphs are weighted connected graphs.

### Theorem 14

1. Every planar graph is strongly 3-path separable [Tho04].
2. Every  $H$ -minor-free graph with  $n$  vertices is strongly  $O(\sqrt{n})$ -separable.
3. There are unweighted  $K_6$ -minor-free graphs with  $n$  vertices for which every strong  $k$ -path separator requires  $k = \Omega(\sqrt{n})$ .

**Proof.** It is known that  $K_r$ -minor-free graphs with  $n$  vertices have treewidth  $O(r^{3/2}\sqrt{n})$  [AST90]. By Theorem 15, we conclude that the graphs excluding a fixed minor of at most  $r$  vertices are strongly  $O(\sqrt{n})$ -separable.

Let  $G$  be the graph composed of a  $t \times t$  mesh augmented with an universal vertex, i.e., a vertex that neighbors all the vertices of the mesh. Since the mesh is  $K_5$ -minor-free,  $G$  is  $K_6$ -minor free. Let  $n = t^2 + 1$  be the number of vertices of  $G$ .

Let  $S$  be any strong  $k$ -path separator for  $G$ . We remark that, since  $G$  has diameter two, any union of  $k$  shortest paths in  $G$  contains at most  $3k$  vertices. In other words,  $|S| \leq 3k$ .

Assume that  $k < t/3$ . Then,  $|S| < t$ , and thus  $S$  cannot intersect all the rows or all the columns of the underlying  $t \times t$  mesh of  $G$ . It follows that there must exist a connected component in  $G \setminus S$  with at least  $\sum_{i=1}^t i = t(t+1)/2$  vertices. Indeed, we can check that, for every  $c \leq t$ , the largest connected component in a  $t \times t$  mesh in which  $c$  vertices have been removed, has at least  $\sum_{i=t-c}^t \max\{i, t\}$  vertices, obtained by removing the diagonal vertices from coordinates  $(c+1, 1)$  to  $(t, c)$ .

Note that  $t(t+1)/2 > (t^2+1)/2 = n/2$  for  $t > 1$ . This is a contradiction with the fact that all connected components of  $G \setminus S$  must have at most  $n/2$  vertices. Therefore,  $k \geq t/3 = \Omega(\sqrt{n})$ .  $\square$

**Theorem 15** Every treewidth- $r$  graph is  $K_{r+2}$ -minor-free and is strongly  $(r+1)$ -path separable. Moreover, there are unweighted treewidth- $r$  graphs for which every  $k$ -path separator requires  $k \geq r/2$ .

**Proof.** Using Lemma 5, every connected graph having a tree-decomposition  $\mathcal{T}$  of width  $w$  has a strong  $(w+1)$ -path separator. So, it suffices to consider a tree-decomposition of optimal width  $r$  for treewidth- $r$  graphs.

Consider now the graph  $G = K_{r, n-r}$  with  $n \geq 2r$ . Its treewidth is  $r$  (cf. [Klo94]), and every set of  $r-1$  vertices does not disconnect  $G$ . Moreover, every shortest path in  $G$  includes at most two

vertices of each stable. Thus, if  $n \geq 2r$ , in order to obtain connected components of size at most  $n/2$  we need to disconnect  $G$ . And, to disconnect  $G$  we need to delete at least  $r/2$  shortest paths.  $\square$

### 5.3 More general separators

Observe that a 3D mesh has no  $k$ -path separators for bounded  $k$ , whereas there exists a “2D mesh separator”. And this graph appears to be an easy graph for solving object location problem. This leads to the following natural extension of the  $k$ -path separability definition.

Recall that a subgraph  $H$  of  $G$  is *isometric* if  $d_H(x, y) = d_G(x, y)$  for all  $x, y \in H$ .  $H$  is of doubling dimension  $\alpha$  if for every radius  $r$  and vertex  $x \in H$ , the radius- $2r$  ball in  $H$  centered at  $x$  can be covered by at most  $2^\alpha$  radius- $r$  balls of  $H$ .

A weighted graph  $G$  with  $n$  vertices is  $(k, \alpha)$ -*doubling separable* if there exists a subset of vertices  $S$ , call  $(k, \alpha)$ -doubling separator, satisfying conditions P2 and P3 of the previous definition and where condition P1 is replaced by:

(P1')  $S = P_0 \cup P_1 \cup \dots$ , where each  $P_i$  is a set of isometric subgraphs of doubling dimension at most  $\alpha$  in  $G \setminus \bigcup_{j < i} P_j$ .

So a  $k$ -path separator is nothing else than a  $(k, 1)$ -doubling separator. Based on  $O((\alpha/\varepsilon)^{O(\alpha)} \log \Delta)$ -bit  $(1 + \varepsilon)$ -approximate labels [Tal04], we can actually show the following:

**Theorem 16** *For every  $n$ -vertex weighted  $(k, \alpha)$ -doubling separable graph  $G$ , and for every  $\varepsilon > 0$ , there exists a data-structure of  $O(\tau \cdot n \log n)$  space supporting  $(1 + \varepsilon)$ -approximate distance queries in  $O(\tau \log n)$  time, where  $\tau \leq k(\alpha/\varepsilon)^{O(\alpha)}$ . Moreover, the data-structure is polynomial time constructible, and can be distributed as a  $(1 + \varepsilon)$ -approximate distance labeling with  $O(\tau \cdot \log n)$  space labels.*

## 6 Conclusion

We showed that weighted graphs excluding a fixed minor have a  $k$ -path separator for some constant  $k$ , i.e., a recursive pruning decomposition in  $k$  minimum cost paths. Our scheme is polynomial time constructible, and the associated data-structures have optimal size up to a  $\log n$  factor, and allow us to solve several object location problems: 1) approximate compact distance oracle, 2) distance labeling, 3) compact routing, and 4) small-worldization. The approximation factor is  $1 + \varepsilon$  for any fixed  $\varepsilon > 0$ .

We generalized our decomposition to separator isometric subgraphs of low doubling dimension in order to capture more sparse graphs. Unfortunately, we have showed that efferent distance oracles cannot be achieved on the class of all sparse  $n$ -vertex graphs (even unweighted graphs).

We leave open several questions:

1) Proves or disprove that  $K_r$ -minor-free graphs are  $k$ -path separable for  $k = r^{O(1)}$ . By Theorem 15,  $k \geq r/2$ , and the best upper bound on  $k$  is greater than  $20^{10^{6r^5}}$  (cf. Section 2.1). A stronger lower bound would be interesting.

2) Is there is a constant  $c > 0$  such that every weighted bounded degree graph is  $(\log^c n, c \log \log n)$ -doubling separable? By [Theorem 16](#), this would imply  $(1 + \varepsilon)$ -approximate solutions with distributed data-structures (labels) of poly-log space for these graphs.

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## A Proof of Lemma 5

**Lemma 5** *There is a bag  $C$  of  $\mathcal{T}$ , call center, such that the connected components of  $G \setminus C$  have at most  $n/2$  vertices.*

**Proof.** We choose an arbitrary bag  $R$  to be the root of  $\mathcal{T}$ , and we denote by  $p(X)$  the parent of the bag  $X$  in  $\mathcal{T}$ . For a bag  $X$ , we denote by  $\mathcal{T}_X$  the subtree of  $\mathcal{T}$  rooted at  $X$ .

Let us define the *weight* of a bag  $X$ , denoted by  $w(X)$ , as the number of vertices of  $G$  contained in  $\mathcal{T}_X \setminus p(X)$ , with the convention  $p(R) = \emptyset$ . Observe that  $w(p(X)) \geq w(X)$  for every bag  $X \neq R$ , and that if  $w(X) \geq n/2$ , then  $\mathcal{T} \setminus \mathcal{T}_X$  contains at most  $n/2$  vertices of  $G$ .

The center  $C$  can be found, traversing  $\mathcal{T}$  from its root  $R$ , as follows:

1. If  $w(F) < n/2$  for all the children  $F$  of the current bag  $X$ , then set  $C = X$  and stop.
2. Otherwise, choose a child  $F$  of  $X$  with  $w(F) \geq n/2$ , set  $X = F$  and go to Step 1.

The above procedure stops after at most the depth of  $\mathcal{T}$  steps. Indeed, at any time along the traversal  $w(X) \geq n/2$ , and  $w(X)$  is non-increasing. Moreover, whenever it stops at  $C$ ,  $w(C) \geq n/2$  and  $w(F) < n/2$  for all the children of  $C$ . In other words, and by definition of the weights,  $\mathcal{T} \setminus C$  is composed of subtrees of  $\mathcal{T}$  containing at most  $n/2$  vertices of  $G$ .  $C$  is the center as claimed.  $\square$

## B Proof of Lemma 6

**Lemma 6** *Let  $G$  be a  $(h, p, g)$ -embedded graph for  $g > 0$ , and let  $T$  be a spanning tree of  $G$  rooted in the embedded part of  $G$ . Then, there exist two vortex-paths  $\mathcal{V}_1, \mathcal{V}_2$  such that each segment is a monotone path of  $T$ , and each connected component of  $G \setminus (\mathcal{V}_1 \cup \mathcal{V}_2)$  is  $(h + 1, p, g - 1)$ -embedded. Moreover,  $\mathcal{V}_1, \mathcal{V}_2$  and the embeddings can be computed in polynomial time.*

**Proof.** Let us denote by  $\Sigma$  the embedding surface of  $G$ , and  $G_\Sigma$  the embedded part of  $G$ .

A simple cycle  $C$  of  $G_\Sigma$  is *surface separating* if  $C$  is twosided and if  $\Sigma \setminus C$  is composed of two regions. Otherwise,  $C$  is said *nonseparating*. It is known that [MT01, pp. 104-106]:

1. If  $C$  is surface separating, then the Euler genus of  $\Sigma$  is the sum of the Euler genera of  $\Sigma_1 \cup C$  and  $\Sigma_2 \cup C$ , where  $\Sigma_1, \Sigma_2$  are the regions of  $\Sigma \setminus C$ .
2. If  $C$  is nonseparating, then  $\Sigma \setminus C$  has Euler genus less than the Euler genus of  $\Sigma$ .

For the ease of presentation, we first describe the construction if  $G$  is  $(h, 0, g)$ -embedded ( $p = 0$ ), i.e., if all the vortex bags of  $G$  reduce to a single vertex, its perimeter. In other words, we are in a quite favorable situation where  $G = G_\Sigma$ . Then, we extend the technique for arbitrary  $p$ .

A cycle  $C$  of  $G_\Sigma$  *simply* intersects a vortex  $W$  if for each side of  $C$  the perimeter vertices of  $W \setminus C$  is a consecutive part of the border face of  $W$ .

**Claim 3** *Let  $C$  be a simple cycle of  $G_\Sigma$  that simply intersects any vortex it encounters. Then, in polynomial time, the connected components of  $G \setminus C$  can be  $(h + 1, 0, g')$ -embedded where  $g'$  is the maximum of the Euler genus over all regions of  $\Sigma \setminus C$ .*

**Proof.** Let  $c$  be the number of vortices  $C$  intersects. We cut  $\Sigma$  along  $C$  and we replace each side of  $C$  by a disc ( $C$  is onesided or twosided depending on the orientability of  $\Sigma$ ). By this way, each side of  $C$  induces a cellular face in the new embedding of  $G \setminus C$ . In particular, the vortices cut by  $C$  all lie on such new face. Because  $C$  simply intersects each vortex, all the  $c$  vortices can be merged into one big vortex, adding artificial empty bags to border vertex if needed. Doing similarly on the other side of  $C$  (if any), we obtain two new vortices in replacement of the vortices cut by  $C$ . In total we get an embedding of each component of  $G \setminus C$  on  $\Sigma \setminus C$  with at most  $h - c + 2 \leq h + 1$  vortices if  $c \geq 1$  (obviously, in case  $c = 0$  there is at most  $h$  vortices). The Euler genus of each connected component of  $G \setminus C$  is at most the one of the corresponding region of  $\Sigma \cap C$ .  $\square$

Therefore, our goal is to apply [Claim 3](#) either on a nonseparating cycle, or on a separating cycle that decreases the Euler genus of its regions.

To prove [Lemma 6](#) (for  $p = 0$ ) we choose an edge  $\{u, v\}$  of  $G_\Sigma \setminus T$ . Such an edge exists since  $G_\Sigma$  cannot reduce to a tree as  $g > 0$ . Let  $C_{uv}$  be the fundamental cycle formed by the nontree edge  $\{u, v\}$  of  $T$ . There are three cases:

**Case 1:**  $C_{uv}$  is nonseparating. By [Claim 3](#), the connected components of  $G \setminus C_{uv}$  can be  $(h + 1, 0, g - 1)$ -embedded.

**Case 2:**  $C_{uv}$  is surface separating, and all the regions of  $\Sigma \setminus C_{uv}$  have nonnull Euler genus<sup>3</sup>. It follows that all the regions have Euler genus  $< g$ , and thus by [Claim 3](#), the connected components of  $G \setminus C_{uv}$  can be  $(h + 1, 0, g - 1)$ -embedded.

**Case 3:**  $C_{uv}$  is surface separating, and there is a region  $\Sigma'$  of  $\Sigma \cap C_{uv}$  such that the Euler genus of  $\Sigma' \cup C_{uv}$  is zero. We can assume that  $G_\Sigma$  is triangulated: since we want to construct a suitable cycle  $C$  in order to apply [Claim 3](#), it does not matter if some edges of  $C$  exists or not in  $G_\Sigma$ , the separability property being preserved.

The edge  $\{u, v\}$  belongs to two triangular faces lying each one on a side of  $C_{uv}$ . Indeed, first  $C_{uv}$  is twosided because it is surface separating, and secondly, if the two faces sharing  $\{u, v\}$  are both in  $\Sigma'$ , they would be included: a contradiction,  $\Sigma'$  is of Euler genus 0.

Let  $w$  be the third vertex of the face  $u, v, w$  that lies outside  $\Sigma'$  ( $w \in C_{uv}$  is possible). We consider now the two more fundamental cycles  $C_{uw}$  and  $C_{vw}$  induced by the nontree edges  $\{u, w\}$  and  $\{v, w\}$  of  $T$ . We note that  $C_{uv}$ ,  $C_{uw}$ , and  $C_{vw}$  have been obtained by combinations of the three tree paths  $T(u, r)$ ,  $T(v, r)$ , and  $T(w, r)$ ,  $r$  the root of  $T$ . Observe that the triangulation of  $G_\Sigma$  serves only to define  $w$ .

We then consider, for each new cycle, the above Case 1 and Case 2. If for both cycles only Case 3 occurs, then all of the three cycles are surface separating with one of the region of Euler genus 0. It follows that a Euler genus 0 region of  $\Sigma \setminus C_{uw}$  (denoted by  $\Sigma_u$ ) or of  $\Sigma \setminus C_{vw}$  (denoted by  $\Sigma_v$ ) includes  $\Sigma'$ . More precisely, either  $\Sigma_u \cup C_{uw} \supset \Sigma' \cup C_{uv}$ , or  $\Sigma_v \cup C_{vw} \supset \Sigma' \cup C_{uv}$ . The new region with its border cycle contains at least one more edge than  $\Sigma' \cup C_{uv}$  (the edge  $\{u, w\}$  or the edge  $\{v, w\}$ ). Therefore, by finiteness of the number of edges of the triangulations of  $G_\Sigma$ , Case 1 or 2 will occur after a polynomial amount of time, completing the proof if  $p = 0$ .

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<sup>3</sup>The Euler genus of any region  $\Sigma'$  of  $\Sigma$  can be computed from the embedding of  $G$  induced by  $\Sigma'$  thanks to the Euler's Formula.

**General case for  $p > 0$ .** Let  $\{u, v\}$  be an edge of  $G_\Sigma \setminus T$ , and consider the vortex-paths  $\mathcal{V}_u$  and  $\mathcal{V}_v$  of the paths  $T(u, r)$  and  $T(v, r)$  respectively. We reorganize the segments of  $\mathcal{V}_u$  and of  $\mathcal{V}_v$  such that the edges of their projections do not intersect in every common vortex. This can be done by exchanging some segments incident to a common vortex as described in Fig. 2(a-b) with a greedy process starting from  $u$  and  $v$ . Observe that this process preserves the monotonicity of the segments: they are still monotone paths of  $T$ .

Let  $C_{uv} = \mathcal{V}_u \cup \mathcal{V}_v$ , and let  $\bar{C}_{uv} = \bar{\mathcal{V}}_u \cup \bar{\mathcal{V}}_v \cup \{u, v\}$  that is the set formed by the projections of  $\bar{\mathcal{V}}_u$  and of  $\bar{\mathcal{V}}_v$  plus the edge  $\{u, v\}$ . See Fig. 2(b). We now assume that  $G_\Sigma$  is triangulated such that all the edges of  $\bar{C}_{uv}$  exist. This is possible since the edges in the projection do not intersect in any of the vortices.

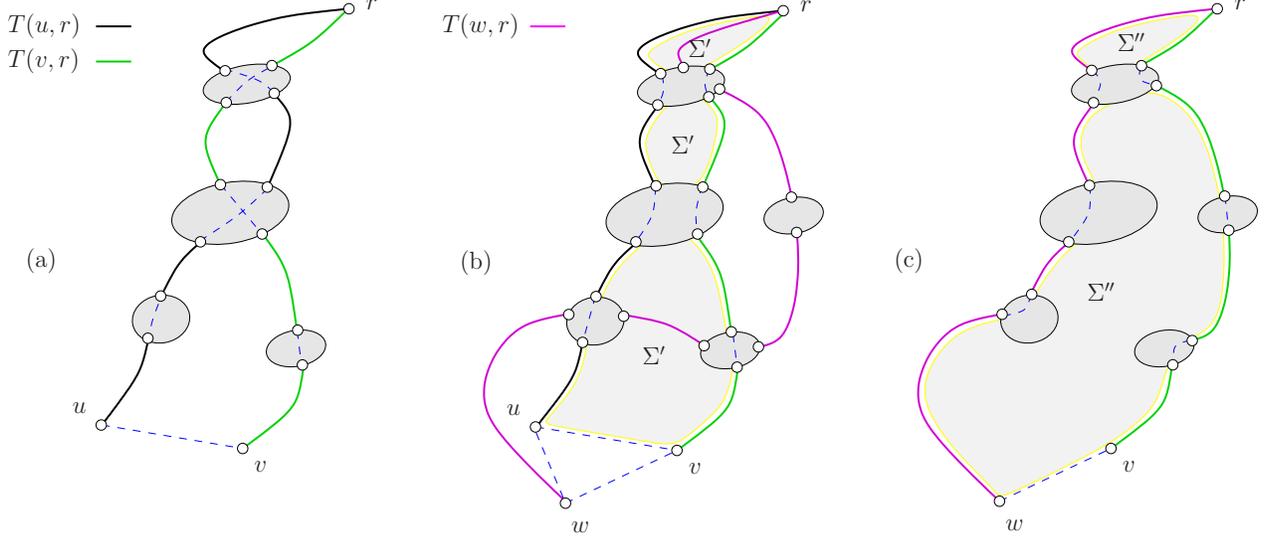


Figure 2: (a) The vortex-paths of  $T(u, r)$  and  $T(v, r)$  with their projections crossing in some common vortices. (b) The vortex-paths  $\mathcal{V}_u$  and  $\mathcal{V}_v$  have been reorganized to avoid their projections intersect. The region  $\Sigma'$  of Euler genus 0 is formed. Vertex  $w \notin \Sigma'$  is added, and also the vortex-path of  $T(w, r)$ . (c) The vortex-paths  $\mathcal{V}_w$  and  $\mathcal{V}_v$  are reorganized with  $\mathcal{V}_u$  and  $\mathcal{V}_v$ . The super region  $\Sigma'' \supset \Sigma'$  is formed increasing its size by at least one new edge:  $\{u, w\}$ .

We observe that  $\bar{C}_{uv}$  induces a simple closed curve of  $\Sigma$ , it is a simple cycle of  $G_\Sigma$ . And,  $\bar{C}_{uv}$  simply intersects all its vortices.  $\bar{C}_{uv}$  concentrates all the properties we need to prove [Claim 3](#) for  $G_\Sigma$ , but not for  $G$ , because some paths in  $G$  may jump over  $\bar{C}_{uv}$  thanks to some vortex edges or vertices that are not in  $G_\Sigma$ . However, every path with extremities in  $G_\Sigma$  that jumps over  $\bar{C}_{uv}$  has to intersect  $C_{uv}$ . This is due to the fact that the vortex bags  $X_i$  and  $Y_i$  of a vortex-path  $\mathcal{V}$  disconnect the vortex in two parts, each part lying on a different side of the edge of  $\bar{\mathcal{V}}$  connecting the perimeter vertices of  $X_i$  and of  $Y_i$ .

Under these conditions, and using the same ideas of [Claim 3](#), we can prove the following (in the next statement,  $\bar{C}$  and  $C$  stands for  $\bar{C}_{uv}$  and  $C_{uv}$  for an arbitrary edge  $\{u, v\}$  of  $G_\Sigma \setminus T$ ):

**Claim 4** *Let  $\bar{C}$  be a simple cycle of  $G_\Sigma$  that simply intersects any vortex it encounters. Then, in polynomial time, the connected components of  $G \setminus \bar{C}$  can be  $(h + 1, p, g')$ -embedded where  $g'$  is the*

maximum of the Euler genus over all regions of  $\Sigma \setminus C$ .

In the General case the three cases are similar to the  $p = 0$  case, and we take only special attention to Case 3:  $\tilde{C}_{uv}$  is surface separating and there is a region  $\Sigma'$  of  $\Sigma \cap \tilde{C}_{uv}$  such that the Euler genus of  $\Sigma' \cup \tilde{C}_{uv}$  is zero. Similarly, we consider a third vertex  $w \in G_\Sigma$  such  $w \notin \Sigma'$ , and its vortex-path  $\mathcal{V}_w$ . We observe that some segments of  $\mathcal{V}_w$  may intersect  $\Sigma'$  thanks to vortices. Obviously, as  $T$  is cycle-free, no two segments of the vortex-paths  $\mathcal{V}_w$  and  $\mathcal{V}_x$ , for  $x = u, v$ , intersect but in the nearest common ancestor between  $w$  and  $x$  in  $T$ . The vortex-path  $\mathcal{V}_w$  is reorganized with  $\mathcal{V}_u$  in order to form the sets  $C_{uw}$  and  $\tilde{C}_{uw}$ . We also reorganize  $\mathcal{V}_w$  with  $\mathcal{V}_v$  to form the sets  $C_{vw}$  and  $\tilde{C}_{vw}$  (note that the curves  $\tilde{C}_{uw}$  and  $\tilde{C}_{vw}$  may result of different segments reorganizations and triangulations of  $G_\Sigma$ , but again, only vertices do matter).

The three cycles  $\tilde{C}_{uv}$ ,  $\tilde{C}_{uw}$ ,  $\tilde{C}_{vw}$  must be surface separating and induced a Euler genus 0 region, since otherwise we can conclude directly with [Claim 4](#). And in this case, one region, say  $\Sigma''$ , must strictly include  $\Sigma'$  (by one more edge) by reorganizing its segments with  $\mathcal{V}_u$  or  $\mathcal{V}_v$  as shown on [Fig. 2\(c\)](#). So, moving from nontree edges to nontree edges in  $G_\Sigma$ , and after a linear number of steps (linear in  $|E(G_\Sigma)|$ ), [Claim 4](#) will be applied. This completes the proof of [Lemma 6](#).  $\square$

## C Proof of [Lemma 7](#)

**Lemma 7**  $T_s$  is a spanning tree of  $\tilde{C}_s$ .

**Proof.** It is clear that  $T_s$  spans  $\tilde{C}_s$ , since  $A_s$  spans  $C_s$  and  $T_s \supseteq A_s \cap C_s$ .

Let us show that  $T_s$  is connected. Choose any two vertices  $u, v \in \tilde{C}_s$ . If  $A_s(u, v) \subseteq T_s$ , then  $u$  and  $v$  are connected in  $T_s$ . So assume  $A_s(u, v)$  has a subpath, say  $A_s(x, y)$ , that goes outside  $\tilde{C}_s$ . W.l.o.g. we can choose  $x$  and  $y$  such that  $A_s(x, y)$  intersects  $\tilde{C}_s$  only in  $x$  and  $y$  and such that every intermediate vertices of  $A_s(x, y)$  does not belong to  $\tilde{C}_s$ . Due to the structure of  $C_s$  in  $G_s$ , the only paths of  $A_s$  that goes outside  $\tilde{C}_s$  have both extremities in a joint set. Because  $r \in \tilde{C}_s$  we conclude  $x$  and  $y$  are related. By the connection rules,  $T_s$  contains the edge  $\{x, y\}$ .

It remains to check that  $T_s$  has no cycle. Assume by contradiction that  $T_s$  contains a cycle  $\mathcal{C}$ .  $\mathcal{C}$  is composed of some edges of the initial tree  $A_s$  (the tree edges), or some edges of the joint sets (the nontree edges).

Let  $\{u, v\}$  be a nontree edge of  $\mathcal{C}$ . Such an edge must exist otherwise  $\mathcal{C}$  would be wholly included in  $A_s$ . If  $\mathcal{C}$  contains only one nontree edge, then  $\mathcal{C}$  is composed of the edge  $\{u, v\}$  plus  $A_s(u, v)$ . There is a contradiction with the fact that  $A_s(u, v) \subset \mathcal{C}$  and  $A_s(u, v)$  is not wholly included in  $\tilde{C}_s$ . So, let  $\{u', v'\}$  be the second nontree edge of  $\mathcal{C}$ . Note that  $u' \neq u$  ( $u' = v$  possibly).

W.l.o.g. assume that  $u$  is ancestor  $v$ , and that walking on  $\mathcal{C}$  from  $v$  to  $u'$  we encounter only tree edges. The path is actually  $A_s(v, u')$ . Every vertex  $w \in A_s(v, u')$  is a descendant of  $v$ , otherwise (if  $w \neq v$ )  $v$  would have two ancestors:  $u$  and  $w$ . We note that  $v'$  is a descendant of  $u'$ , otherwise  $u'$  would have two ancestors:  $v'$  and  $v$ . In particular  $u$  is ancestor of  $u'$  and  $v'$  with  $u \neq u'$ . Similarly, the nontree edge  $\{u'', v''\}$  next to  $\{u', v'\}$  satisfies that  $u'$  is ancestor of  $u''$  and of  $v''$  with  $u'' \neq u'$ .

More generally, any nontree edge  $\{x, y\}$  of  $\mathcal{C}$ , where  $x$  is encountered before  $y$  walking on  $\mathcal{C}$  from  $u$  to  $v$ , have the following property:  $u$  is ancestor of  $x$  and of  $y$  with  $u \neq x$ . Now, starting the walk on  $\mathcal{C}$  from  $x$  to  $y$ , i.e., exchanging the roles of the ordered pair  $(u, v)$  and  $(x, y)$ , we get that  $x$  is

ancestor of  $u$  and  $v$  with: a contradiction, we just proved above that  $u$  is ancestor of  $x$  with  $u \neq x$ .  $\square$

## D Proof of Lemma 8

**Lemma 8** *If  $C_s \cap C_{\Sigma_0} = \emptyset$ , then there was a vortex bag  $X$  of  $C_0$  such that the connected components of  $G_s \setminus X$  have at most  $n/2$  vertices.*

**Proof.** In the torso  $\tilde{G}_s$ , any joint set between  $\tilde{C}_s$  and the components of  $\tilde{G}_s \setminus \tilde{C}_s$  is a clique, say  $K$ , that must be  $h$ -almost embeddable. As  $C_s \cap C_{\Sigma_0} = \emptyset$ ,  $K$  does not contain any vertices on the surface. In particular  $K$  does not contain any perimeter vertices. It follows that  $K$  is included in a unique vortex, say  $W_i$ , otherwise there would exist an edge between two vortices, that is possible only between perimeter vertices. Moreover, as  $G_s$  is connected,  $C_s$  reduces to  $W_i$ . Indeed, since vortices are disjoint, every path from  $W_i$  to  $W_j$  with  $j \neq i$  must use some edge of  $G_s \setminus C_s$ . This implies some edges between  $W_i$  and  $W_j$  in  $\tilde{C}_s$  that is again not possible as there are no perimeter vertices. From tree-decomposition definition, every clique is necessarily contained in a bag of the decomposition. In particular, the path-decomposition of the vortex  $W_i$  has a bag containing  $K$ .

The tree-decomposition  $\mathcal{T}_s$  of  $G_s$  can be reorganized by replacing the bag  $C_s$  with the path-decomposition of  $W_i$ . This forms a tree-decomposition  $\mathcal{T}'_s$  of  $G_s$  since we have seen that each joint set is included in some vortex bag of  $W_i$ . We choose one extremity of the path-decomposition as the root of  $\mathcal{T}'_s$ , and this root bag, say  $R$ , is augmented with  $n - |V(G_s)|$  independent vertices so that  $\mathcal{T}'_s$  is a tree-decomposition of a graph with exactly  $n$  vertices.

We can now reapply the technique of Lemma 5, and we check that, since the weight (as defined in the proof of Lemma 5) of the components out of the path-decomposition of  $W_i$  is  $\leq n/2$ , the walk will be constrained to stay on the path-decomposition of  $C_s$ . Therefore, a bag of  $W_i$ , say  $X$ , will disconnect  $G_s$  in connected components of  $\leq n/2$  vertices as claimed.  $\square$

## E Proof of Lemma 9 and Lemma 10

**Lemma 9** *Let  $C$  be a center of tree-decomposition of an  $n$ -vertex graph  $G$ . Then, in polynomial time, one can construct a clique-weight  $(\mathcal{K}, \omega)$  for the torso  $\tilde{C}$  such that, given any half-size separator  $S$  of  $\tilde{C}$ , the connected components of  $G \setminus S$  have at most  $n/2$  vertices.*

**Proof.** Let  $\mathcal{T}$  be the tree-decomposition of  $G$ . The clique-weight  $(\mathcal{K}, \omega)$  for  $C$  is defined as follows:  $\mathcal{K}$  is the set of all the single vertex clique (each vertex of  $C$  being considered as a clique) plus all the joint sets of  $\tilde{C}$  in  $\mathcal{T}$ . And, for every vertex  $u \in C$ ,  $\omega(\{u\}) = 1$ , and for every joint set  $K \in \mathcal{K}$  joining  $\tilde{C}$  to the subgraphs  $B_1, \dots, B_t$ , set  $\omega(K) = \sum_{i=1}^t |B_i|$ .

Let  $\mathcal{B}$  be the set of all the components of  $G \setminus C$ . Summing over the two types of cliques (the vertices and the joint sets of  $\tilde{C}$ ), we obtain:

$$f(C) = \sum_{\substack{K \cap C \neq \emptyset \\ K \in \mathcal{K}}} \omega(K) = |C| + \sum_{B \in \mathcal{B}} |B| = n$$

because  $\{C\} \cup \mathcal{B}$  is a partition of  $V(G)$ .

Consider now any  $S \subseteq C$ , and let  $A$  be a connected component of  $G \setminus S$ .

If  $C \cap A = \emptyset$ , then it follows that  $A$  is a connected component of  $G \setminus C$ , and thus  $|A| \leq n/2$  since  $C$  is the center of  $\mathcal{T}$ .

If  $C \cap A \neq \emptyset$ , then let  $x \in C \cap A$  and  $y \in C$ . Let us show that there is a path in  $\tilde{C} \setminus S$  from  $x$  to  $y$  if and only if  $y \in C \cap A$ . In other words, let us show that  $\tilde{C} \cap A$  is a connected component of  $\tilde{C} \setminus S$ . Clearly, if  $y \in C \setminus A$ , then, every path from  $x$  to  $y$  must intersect  $S$  (because  $A$  is a connected component of  $G \setminus S$ ). Conversely, if  $y \in C \cap A$ , by the connexity of  $A$  in  $G \setminus S$ , there is a path  $P$  in  $G$  from  $x$  to  $y$ . Since both extremities of  $P$  are in  $C \cap A$ , if  $P$  leaves  $C$  through some joint set  $K$ , then  $P$  must go back to  $C$  through the same  $K$ . In particular, using the first vertex (from  $x$ )  $u \in P \cap K$  and the last vertex  $v \in P \cap K$ , we can shortcut  $P$  thanks to the edge  $\{u, v\}$  of  $K$ . This edge is in  $\tilde{C} \cap A$  as  $u, v$  both are in  $C \cap A \cap K$ . Repeating this process on the shortcut, one can construct a path from  $x$  to  $y$  wholly included in  $\tilde{C} \cap A$ . Therefore,  $\tilde{C} \cap A$  is a connected component of  $\tilde{C} \setminus S$ .

An important observation is that if a clique  $K$  of  $\tilde{C}$  intersects  $A$  and  $u \in K$ , then  $u \in A \cup S$ . Otherwise, as  $K$  is a clique, there would have an edge from  $u \notin A \cup S$  to a vertex of  $A$  contradicting the fact that  $A$  is a connected component of  $G \setminus S$ . Let  $C' = \tilde{C} \cap A$  be the connected component, and let  $\mathcal{B}' = \{B' \in \mathcal{B} \mid B' \cap C' \neq \emptyset\}$ . From the previous remark,  $|A| \leq |C'| + \sum_{B' \in \mathcal{B}'} |B'|$ . By construction of the clique-weight,  $|C'| + \sum_{B' \in \mathcal{B}'} |B'| = f(C')$ . Now if  $S$  is a half-size separator for  $\tilde{C}$ , so  $f(C') \leq n/2$ , proving that  $|A| \leq n/2$  as claimed.  $\square$

**Lemma 10** *Let  $G$  be a nearly planar graph with a clique-weight and a spanning tree  $T$  rooted in the planar part of  $G$ . In polynomial time, one can construct a half-size separator for  $G$  composed of at most three vortex-paths whose segments consist of monotone paths of  $T$ .*

**Proof.** This proof has similarity with the last part of the proof of [Lemma 6](#). It is also inspired from [\[Tho04\]](#) and [\[LT79\]](#).

Let  $\sigma = f(G)$  be the weight of  $G$  having a clique-weight  $(\mathcal{K}, \omega)$ . Note that  $\sigma \geq 0$  since weights are positive. Let  $G_{\mathbb{S}}$  be the planar part of  $G$ ,  $n = |V(G_{\mathbb{S}})|$ , and let  $r \in G_{\mathbb{S}}$  be the root of  $T$ .

We assume that  $n \geq 3$ , otherwise  $G_{\mathbb{S}}$  is a trivial half-separator for  $G$  composed of at most three vortex-paths. Any triangulation of  $G_{\mathbb{S}}$  has  $3n - 6$  edges. We observe that every closed curve  $\mathcal{C}$  of the sphere  $\mathbb{S}$  is twosided:  $\mathbb{S} \setminus \mathcal{C}$  is composed of two *regions*. For convenience, a region is either a connected part of  $\mathbb{S}$ , or the subgraph of  $G_{\mathbb{S}}$  embedded on this part. If  $R$  is a region, then we denote by  $R^*$  its complement region. Formally, for any region  $R$  with border  $\mathcal{C}$ ,  $R^* = \mathbb{S} \setminus (R \cup \mathcal{C})$ . Since no cliques of  $\mathcal{K}$  can intersect both  $R$  and  $R^*$ , it follows that  $f(R) + f(R^*) \leq \sigma$ , for any region  $R$ .

Let  $u, v$  be two vertices of  $G_{\mathbb{S}}$  lying on a same border face. If  $G_{\mathbb{S}} = T$ , then any pair is fine, otherwise one can choose an edge of  $G_{\mathbb{S}} \setminus T$ .

We construct the vortex-paths  $\mathcal{V}_u$  and  $\mathcal{V}_v$  of  $T(u, r)$  and  $T(v, r)$  respectively. We assume that  $\mathcal{V}_u$  and  $\mathcal{V}_v$  have been reorganized as done in the proof of [Lemma 6](#) such that their projections do not intersect inside the face of any common vortex. From the projections of the two vortex-paths, we let  $\bar{C}_{uv} = \bar{\mathcal{V}}_u \cup \bar{\mathcal{V}}_v \cup \{u, v\}$ . Note that, once reorganized the projections  $\mathcal{V}_u$  and  $\mathcal{V}_v$ , there exists a triangulation of  $G_{\mathbb{S}}$  containing  $\bar{C}_{uv}$  as a cycle.

Let  $A$  be a region of  $G_{\mathbb{S}} \setminus \bar{C}_{uv}$ . For illustration,  $A$  corresponds to the region  $\Sigma'$  depicted on [Fig. 2\(b\)](#). W.l.o.g., assume  $f(A) \leq \sigma/2$ . If  $f(A^*) \leq \sigma/2$ , then we can stop and return the two current vortex-paths  $\mathcal{V}_u$  and  $\mathcal{V}_v$ . So assume that  $f(A^*) > \sigma/2$ . Note that  $f(A^*) > 0$ .

Because  $f(A^*) > 0$  and  $\omega(K) \geq 0$  for every  $K \in \mathcal{K}$ , it implies that  $A^* \neq \emptyset$ , and so there

exists a vertex  $w \in A^*$  such that  $u, v, w$  lie on the border of the same face of  $G_{\mathbb{S}}$ . We construct the vortex-path  $\mathcal{V}_w$  of  $T(w, r)$ .  $\mathcal{V}_w$  can intersect  $A$  through some common vortex with  $\mathcal{V}_u$  or  $\mathcal{V}_v$ . If it occurs, then  $\mathcal{V}_w$  is merged with  $\mathcal{V}_u$  on its part going from the first common vortex up to  $r$  (it can be merged with  $\mathcal{V}_v$  depending on which vortex-path it intersect first).

Similarly to  $\bar{C}_{uv}$ , we define the two extra curves  $\bar{C}_{uw} = \bar{\mathcal{V}}_u \cup \bar{\mathcal{V}}_w \cup \{u, w\}$ , and  $\bar{C}_{vw} = \bar{\mathcal{V}}_v \cup \bar{\mathcal{V}}_w \cup \{v, w\}$ , and we augment  $G_{\mathbb{S}}$  with the three edges of the triangle  $u, v, w$ .

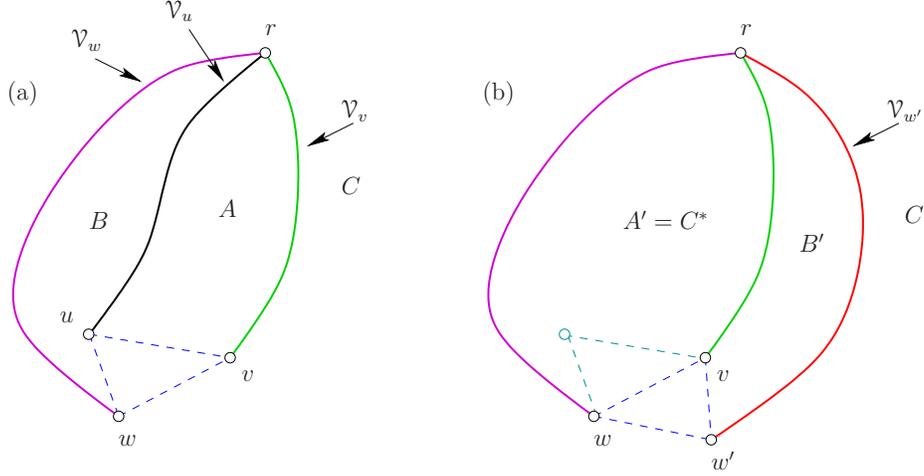


Figure 3: (a) The vortex-paths  $\mathcal{V}_u$ ,  $\mathcal{V}_v$ , and  $\mathcal{V}_w$ . (b) Finding the next triangle in the heaviest region.

At this step (cf. Fig. 3(a)), we have the three curves defining three regions:  $A$ ,  $B$ , and  $C$ , where  $B \cup C \subset A^*$ . Note that one region of the triangle  $u, v, w$  has no vertices, and so has null weight. W.l.o.g. assume  $f(C) \geq f(B)$ .

If  $f(C) \leq \sigma/2$ , then the three vortex-paths form the half-separator of  $G$ , since  $f(B) \leq f(C) \leq \sigma/2$  and  $f(A) \leq \sigma/2$ .

Thus region  $C$ , whose border is  $\bar{C}_{vw}$  (or  $\bar{C}_{uw}$ ), verifies that  $f(C) > \sigma/2$ . Merging regions  $A$  and  $B$  into the region  $A' = C^*$ , we obtained that  $A'$  is a region of  $G_{\mathbb{S}} \setminus \bar{C}_{vw}$  and that  $f(A'^*) = f(C) > \sigma/2$ . Note that  $A'$  includes  $A$  and contains at least one edge more than  $A$ : the edge  $\{u, w\}$  or  $\{v, w\}$ .

From the above argument ( $f(A'^*) > 0$  and  $\omega(K) \geq 0$ ) there is a vertex  $w' \in A'^*$  lying on the same border face of  $v$  and  $w$  (cf. Fig. 3(b)). Iterating the process, one constructs a third vortex-path  $\mathcal{V}_{w'}$ , and obtains three regions  $A'$ ,  $B'$ , and  $C'$  by selecting each time the next vertex in the heaviest region. Clearly, if the process ends, we are done with two or three regions (and their corresponding vortex-paths) of weight at most  $\sigma/2$ .

The process eventually ends because each time we merge regions  $A$  and  $B$  into  $A'$ ,  $A'$  includes at least one more edge, so it ends after at most  $O(n)$  steps, completing the proof.  $\square$

## F Proof of Theorem 3

We use an addition/comparison model of computation, that is, the edge-weights can be arbitrary real, non-negative values and we only compare values that come from sums of edge-weights. In that

model, one unit of space consists of a block of  $\Omega(\omega + \log n)$  bits of memory, where  $n$  is a number of vertices in the graph and  $\omega$  is the number of bits to represent an edge-weight. In particular, for integral edge-weights, a block has length  $\Omega(\log \Delta)$  bits where  $\Delta$  is the weighted diameter of the graph.

Given a weighted graph  $G$  and  $u, v \in V(G)$ , let  $d_G(u, v)$  be the cost of a minimal cost path in  $G$  between  $u$  and  $v$ . We omit the subscript  $G$  when it is clear from the context. Distance is extended naturally to sets, given sets  $U, V \subseteq V(G)$ , let  $d(U, V) = \min \{d(u, v) \mid u \in U, v \in V\}$ .

Let  $G$  be a weighted  $n$  node  $k$ -path separable graph with aspect ratio  $\Delta = \max d(u, v) / \min d(u, v)$ . To avoid dragging a normalization constant we assume  $\min d(u, v) = 1$ . For any subgraph  $H$  of  $G$  let  $S(H)$  be the set of sets of paths that separate  $H$  in to small components.

We build a rooted tree  $\mathcal{T}$  called the *decomposition tree* of  $G$  as follows. The nodes of  $\mathcal{T}$  are some subgraphs of  $G$ . The root node of  $\mathcal{T}$  is  $G$ . For any  $H \in V(\mathcal{T})$  its children  $J_1, \dots, J_t$  are the connected components induced by  $H \setminus S(H)$ . Observe that the depth of  $\mathcal{T}$  is at most  $\log n$ .

For each  $H \in V(\mathcal{T})$  and each  $v \in H$  and each  $Q \in S_i \in S(H)$ , let  $L(v, Q)$  be the set of landmarks obtained from Thorup [Tho04, Lemma 3.17] on the graph  $H \setminus \bigcup_{j < i} P_j$ ,  $|L(v, Q)| = O(1/\varepsilon)$ . Each node records the name of each node in  $L(v, Q)$  and its distance. This requires  $O(k/\varepsilon \cdot \log n)$  space label per node. The total memory of all labels is  $O(k/\varepsilon \cdot n \log n)$ .

Given any  $s, t \in V(G)$ , let  $H \in V(\mathcal{T})$  be the farthest component from the root that contains a shortest path  $R$  between  $s$  and  $t$ .

From all the paths in  $S(H)$  that intersect  $R$ , let  $Q$  be one with a minimal  $i$  such that  $Q \in P_i \in S(H)$ . Let  $J$  be the graph  $H \setminus \bigcup_{j < i} P_j$ . Hence  $d_J(s, t) = d(s, t)$ . Let  $a \in Q$  be the first node of  $R$  intersecting  $Q$ , let  $b \in Q$  be the last node in  $Q$  intersecting  $R$ . In a fashion similar to [Tho04, Theorem 3.19], the distance is  $1 + \varepsilon$  approximated by the distance from  $s$  to the node in  $L(s, Q)$  covering  $a$  plus the distance between  $a$  and  $b$  plus the distance between the node in  $L(t, Q)$  covering  $b$  to  $t$ .

In the full paper we will show how to extend this approach to all the problems considered in [Tho04]. Specifically to labeled routing and to reachability in directed graphs.