Ratioquadrics: An Alternative Model for Superquadrics

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Abstract
This paper presents a new family of 2D curves and its extension to 3D surfaces, respectively called *ratioconics* and *ratioquadrics*, that have been designed as alternatives to the well-known superconics and superquadrics. This new model is intended to improve the original one on three main points : first it involves lower computation cost and provides better numerical robustness, second it offers higher order continuities (C^1/G^2 or C^2/G^2 instead of C^0/G^0), and third it provides a greater variety of shapes for the resulting curves and surfaces. All these improvements are obtained by replacing the signed power function involved in the formulation of superconics and superquadrics by linear or quadratic rational polynomials.
Keywords: *Geometric Modeling, Parametric Surfaces, Implicit Surfaces, Superquadrics.*

1 Introduction

Since their introduction, superconics and superquadrics, respectively developped by P. Hein [6] and A. Barr [2], have been widely recognized as a powerful and elegant mathematical model. Their properties have been intensively studied [3] and they have been successfully employed in many different application fields. For instance, superconics (and more precisely, superellipses) have become a model of choice for automatic reduction of statistical or experimental 2D data, mainly because the number of their degrees of freedom is small and the role of these parameters is easily predictible. The same data-fitting process has also been used for superquadrics and 3D data [12, 7, 11, 1]. In the field of computer-aided design the major limitation of the basic model (i.e. it does not allow free-form modeling) has been largely reduced over the years : either by using superquadrics as field functions in the *blob* model [5, 14, 8], by employing powerful deformation techniques [10], or by defining a spline-based representation of superquadrics [13]. For all these reasons, superconics and superquadrics represent a basic set of geometrical primitives for 2D and 3D computer graphics that are present in many modeling environments.

In this paper, we present alternative models called *ratioconics* and *ratioquadrics*, which are intended to improve the previous ones on several points (computation cost, numerical robustness, order of parametric and geometric continuity, variety of resulting shapes). More precisely, the article is organized as follows : Section 2 recalls the basics of superconics and superquadrics, Section 3 (respectively Section 4) presents the new model in its parametric (respectively implicit) form, and finally Section 5 proposes some extensions.

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2 Superconics and Superquadrics

2.1 Superconics

Let us recall the basics of superconics by taking the simple example of the supercircle (but remember that all other superconics are obtained in a similar way by adding scaling factors and using different trigonometric functions, see [2] for additional details). Starting from the familiar parametric equation of the unit circle :

$$\forall u \in [-\pi, \pi] \qquad \begin{cases} x = \cos u \\ y = \sin u \end{cases} \tag{1}$$

the unit supercircle is obtained by :

$$\forall u \in [-\pi, \pi] \qquad \begin{cases} x = f_p(\cos u) \\ y = f_p(\sin u) \end{cases} \tag{2}$$

where $f_p(t)$ is the *signed power function* (see Figure 1) :

$$\forall t \in [-1, 1] \qquad f_p(t) = \text{sign}(t) \quad |t|^p \tag{3}$$

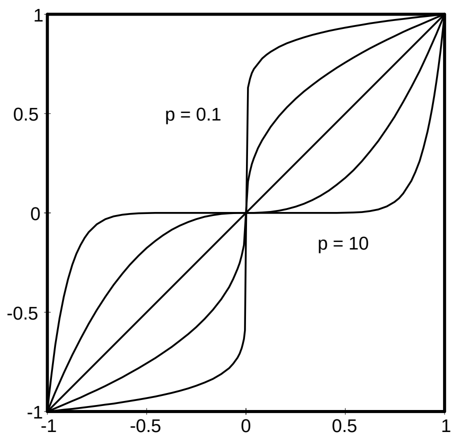
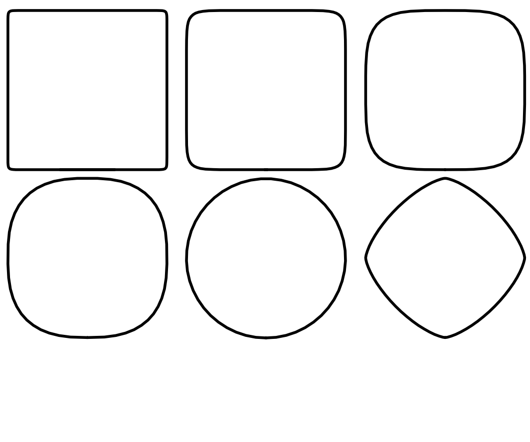


Figure 1 : $f_p(t)$ *for* $p = 0.1, 0.4, 1, 2.5, 10$

Parameter p in Equation 2 is a degree of freedom that must remain positive. As illustrated in Figure 2, this parameter controls the shape of resulting curve in the following way :

- $p = 1$: the curve is a circle
- $p < 1$: the curve becomes more and more "squary"
- $p > 1$: the curve becomes more and more "pinchy"



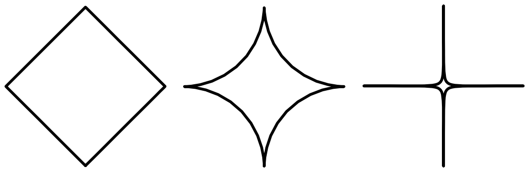


Figure 2 : *Supercircles (from upper-left corner to lower-right corner,* $p = 0.05, 0.1, 0.5, 0.75, 1, 1.5, 2, 3, 9$*)*

2.2 Superquadrics

Superquadrics can be seen as a 3D extension of superconics, obtained by a spherical product of two superconics [2]. For instance, the product of a circle by an orthogonal half-circle provides a superspheroid :

$$\begin{array}{l} \forall u \in [-\pi, \pi] \\ \\ \forall v \in [-\pi/2, \pi/2] \end{array} \qquad \left\{ \begin{array}{l} x = f_p(\cos u)\ f_q(\cos v) \\ y = f_p(\sin u)\ f_q(\cos v) \\ z = f_q(\sin v) \end{array} \right. \tag{4}$$

and the product of two orthogonal circles (one of them being offset by $r > 1$) provides a supertoroid :

$$\begin{array}{l} \forall u \in [-\pi, \pi] \\ \\ \forall v \in [-\pi, \pi] \end{array} \qquad \left\{ \begin{array}{l} x = f_p(\cos u)\ (f_q(\cos v) + r) \\ y = f_p(\sin u)\ (f_q(\cos v) + r) \\ z = f_q(\sin v) \end{array} \right. \tag{5}$$

This time, the equations include two degrees of freedom, a west-east shape parameter p and a north-south shape parameter q (both of them must remain positive). As illustrated in Figure 3 and Figure 4, many different shapes can be obtained, according to the values given to these parameters.

3 Ratioconics and Ratioquadrics

Despite their nice properties, superconics and superquadrics have also three serious weaknesses. First, the computation of the signed power function $f_p(t)$ that is involved in their formulation is relatively expensive. Second, when p becomes small or large (relative to 1) the high steepness or flatness of the function in the neighbourhood of $t = 0$ creates numerical imprecisions when generating the curves or surfaces. Third, when $p < 1$, the first derivative of $f_p(t)$ becomes null at $t = 0$, which leads to the creation of singularities on the resulting shapes. At these singularities, the continuity is only C^0/G^0 and the parametrization is pretty bad. Consequently, the “pinchy” members of the family ($p > 1$ and/or $q > 1$) are seldom used in practice.

In fact, all these unsatisfactory points can be solved simultaneously by replacing the signed power function by another one. We have shown elsewhere [9] that there exists a linear rational polynomial which is a nice alternative to the power function in the range $[0, 1]$. This function can be extended to the range $[-1, 1]$ with a similar construction as the signed power function, giving a *signed linear rational function* :

$$\forall t \in [-1, 1] \qquad g_p(t) = \mathrm{sign}(t)\ \frac{|t|}{p + (1-p)|t|} \tag{6}$$

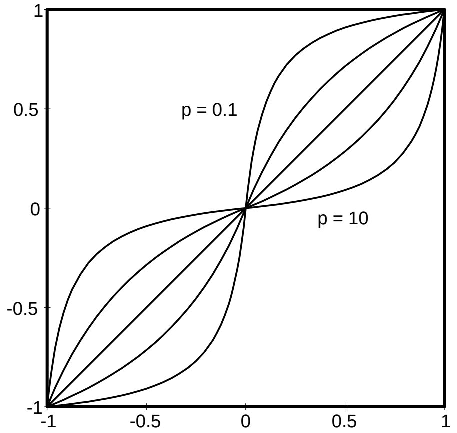


Figure 5 : $g_p(t)$ *for* $p = 0.1, 0.4, 1, 2.5, 10$

Compared to $f_p(t)$, the computation of $g_p(t)$ is very inexpensive (an optimized implementation, which precomputes and stores the value of $1-p$, requires only 1 addition, 1 multiplication and 1 division). The two other interesting properties of $g_p(t)$ for our goal are, first, the derivative that is always strictly positive whatever the value of p, and second, the parametrization that is much more uniform as with $f_p(t)$.

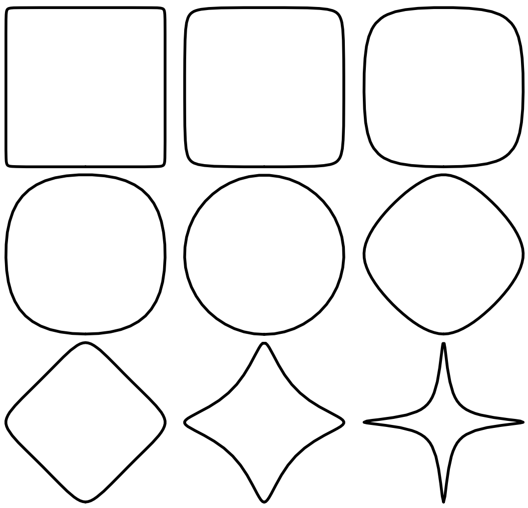


Figure 6 : *Linear Ratiocircles (same p as Figure 2)*

Once the function $g_p(t)$ has been defined, the construction of the corresponding *linear ratioconics* and *linear ratioquadrics* is straightforward: the only thing to do is to replace f_p by g_p in the previous expressions (Equations 3, 4 and 5). The resulting curves and surfaces — call them linear ratiocircles, linear ratiospheroids and linear ratiotoroids — are shown in Figures 6, 7 and 8. The "squary" members of the new family are very close to the old ones, but the "pinchy" members are quite different. The most noticeable fact is that all the sharp edges have been removed (even for large values of the shape parameters) but more precisely, one can show that the continuity of the curves and surfaces is now C^1/G^2, instead of C^0/G^0 for ratioconics and ratioquadrics. Here is the derivation of this latter statement, in the case of ratiocircles :

Let $C(u) = (g_p(\cos u), g_p(\sin u))$ be a ratiocircle of parameter p. The only positions where there may be a discontinuity in the derivatives of $C(u)$ are the points where the sign of $\sin u$ or $\cos u$ changes. For symmetry reasons, we can limit our study to the point $C(0) = (1, 0)$. If we call O^+ (respectively O^- the right (respectively left) limit at $u = 0$, we obtain :

$$C'(O^-) = \left(0\ ,\frac{1}{p}\right) \qquad C''(O^-) = \left(-p\ ,\frac{2\ (1-p)}{p^2}\right)$$
$$C'(O^+) = \left(0\ ,\frac{1}{p}\right) \qquad C''(O^+) = \left(-p\ ,\frac{-2\ (1-p)}{p^2}\right)$$

Obviously, the continuity of $C(u)$ is C^1 but not C^2. To test the G^2 continuity, we have to check if the "curvature vector" $K(u)$ is continuous :

$$K(u) = \frac{\|C'(u) \times C''(u) \times C'(u)\|}{\|C'(u)\|^4} \tag{7}$$

At point $u = 0$, we obtain :

$$K(O^-) = \left(-p^3, 0\right) \qquad K(O^+) = \left(-p^3, 0\right)$$

which proves the G^2 continuity.

4 Implicit Equations

A major characteristic of superconics and superquadrics is that they have also a very simple implicit formulation in addition to the parametric one. For instance, the implicit equation of the unit supercircle is given by :

$$f_{1/p}(x^2) + f_{1/p}(y^2) = 1 \tag{8}$$

and the implicit equation of the unit superspheroid by :

$$f_{p/q}\left(f_{1/p}(x^2) + f_{1/p}(y^2)\right) + f_{1/q}(z^2) = 1 \tag{9}$$

Having such a double formulation is a major benefit in the field of computer graphics because it offers the largest range of visualization algorithms: projective techniques would generally use the parametric formulation (which enables easy tessellation), whereas ray tracing techniques would prefer the implicit one (which can be used as an inside/outside function for ray intersections).

Unfortunately, ratioconics and ratioquadrics do not have such an equivalent implicit formulation. Nevertheless, when replacing f by g in Equation 8 and Equation 9, one obtains implicit curves and surfaces that are very close to the corresponding parametric ones (compare Figure 9 and Figure 6 for instance).

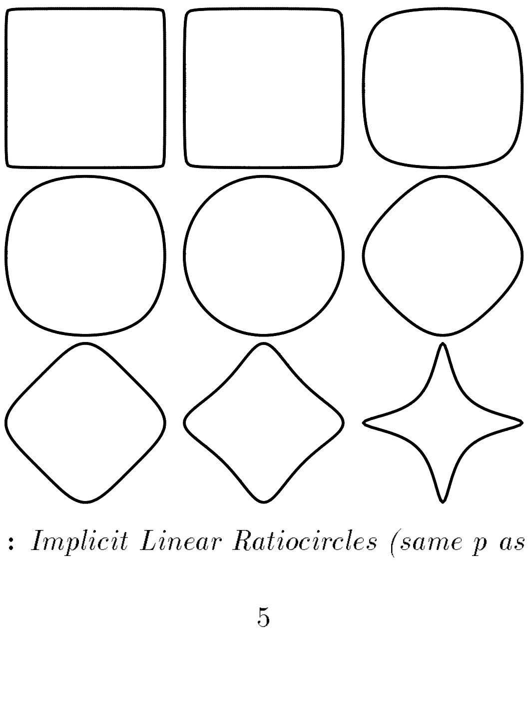


Figure 9 : *Implicit Linear Ratiocircles (same p as Figure 2)*

When displaying the parametric and the implicit curves together on the same plot, it becomes clear that they are not exactly the same, especially for the "pinchy" members (see Figure 10). A numerical estimator of the difference between the two models can be obtained by computing the relative deviation ε of the area bounded by each curve :

p	0.2	0.5	2	5
ε	1.18%	1.45%	7.56%	36.3%

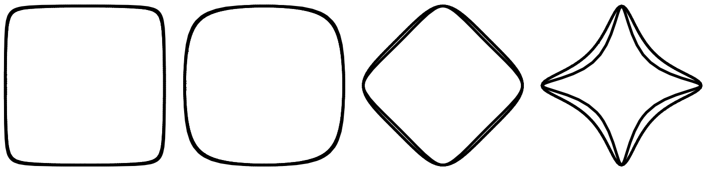


Figure 10 : *Comparison between parametric and implicit ratiocircles (from left to right, $p = 0.2, 0.5, 2, 5$)*

Several comments can be made on this particular point :

- The reason for which implicit superconics/superquadrics are identical to parametric superconics/superquadrics comes from two specific properties of function $f_p(t)$:

$$\forall t \in [-1,1] \qquad f_p(t) \circ f_{1/p}(t) = t$$

$$\forall (u,v) \in [-1,1]^2 \qquad f_p(uv) = f_p(u)\ f_p(v)$$

  Function $g_p(t)$ fulfills the first property *exactly*, but the second property *only approximatively;* this explains why implicit and parametric ratioconics/ratioquadrics are close but not equivalent.

- The implicit ratioconics/ratioquadrics with a given value for p do not represent the best approximation to the parametric ratioconics/ratioquadrics with the same value p. By comparing Figure 9 and Figure 6, one can notice that for the implicit model the "pinchness" grows slower than for the parametric one. In fact, some error minimization technique using a reasonable metric (e.g. area deviation) could be used to find value p' of the implicit model that approaches at best the parametric model with value p. For instance, in the case of the ratiocircle, a simple fitting process which does nothing more than equating the points that are on the first diagonal (for which the deviation is maximal), provides pretty good results with a very simple formulation :

  Let $D(x,y)$ and $D'(x',y')$ be the intersection points of the ratiocircles (respectively parametric and implicit) with the first diagonal. We have

$$x = y = \frac{1}{1 - p + \sqrt{2}\ p} \qquad \text{and} \qquad x' = y' = \sqrt{\frac{1}{1+p'}}$$

  which means that the two points are equal when $p' = (3 - 2\ \sqrt{2})\ p^2 + (2\ \sqrt{2} - 2)\ p$. Figure 11 shows again the parametric and the implicit curves on the same plot, but this time the corrected shape parameter has been used for the implicit one. As shown on the table below, there is less than a 1% difference between the area bounded by the corrected implicit curve and the area bounded by the corresponding parametric one.

p	0.2	0.5	2	5
ε	0.19%	0.29%	0.63%	0.88%

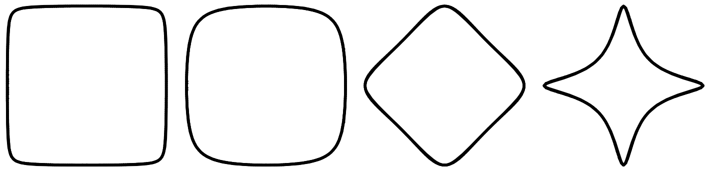


Figure 11 : *Comparison between parametric and corrected implicit ratiocircles (from left to right, $p = 0.2, 0.5, 2, 5$ and p' computed accordingly)*

Anyway, there are very few applications where the parametric and the implicit expressions are needed simultaneously; therefore the difference between the resulting curves and surfaces is unlikely to be damageable in practice. What is important is that there *is* an implicit formulation for ratioquadrics and consequently these surfaces can be easily included in ray-tracing environments. Moreover, the resulting inside/outside function (a rational polynomial with no pole in the definition range) has very good numerical stability and allows efficient root bracketing algorithms (e.g. Sturm) for the ray/surface intersections.

But undoubtedly, the most important benefit of having an implicit formulation is that it allows the user to combine several ratioquadrics by using the *blob* model, and therefore to create much more complex and various shapes. The blob model has been originally developped by Blinn [5]. Its basic idea is to define some points of the space as sources of a potential field and then to compute the 3D isosurface of this field for a given threshold value. Blinn used spherical fields in his paper, but an extension to superquadric fields has been proposed by Wyvill [14]. Let us briefly recall the principle of this extension and show how it can be adapted to ratioquadrics.

To use the blob model, one must be able to compute the value of the field generated by one simple source (that can be placed at the origin without loss of generality) at any point $P(x, y, z)$ of the space. The solution proposed in [14] can be divided in two steps. First, one has to find the point $\overline{P}(\overline{x}, \overline{y}, \overline{z})$ which is the intersection between the superquadric that defines the shape of the field and the line which links the origin O and point P :

- $\overline{P}$ is on the line OP means

$$\exists d > 0 \;\; / \;\; (x, y, z) = (d\,\overline{x}, d\,\overline{y}, d\,\overline{z}) \tag{10}$$

- $\overline{P}$ is on the superspheroid means

$$f_{p/q}\left(f_{1/p}(\overline{x}^2) + f_{1/p}(\overline{y}^2)\right) + f_{1/q}(\overline{z}^2) \tag{11}$$

In fact, the only value needed for the blob model is the squared distance ratio d^2 between P and $\overline{P}$. Equation 10 and Equation 11 are sufficient to compute this value :

$$d^2 = \left[\left(x^{2/p} + y^{2/p}\right)^{p/q} + z^{2/q}\right]^q \tag{12}$$

In the second step, d^2 passes through a potential function $\Phi(d)$ that provides the final value of the field at point P. For instance, one can use the following function (that we have presented in [4]), which merges many useful properties :

$$\Phi(d) = (d^2 < 1)\ ?\ (d^2 < \frac{1}{4})\ ?\ 1 - \frac{(3d^2)^2}{p + (4.5-4p)d^2}\ :\ \frac{(1-d^2)^2}{0.75-p+(1.5+4p)d^2}\ :\ 0 \tag{13}$$

The whole process is relatively expensive. The main cost comes from Equation 12 that involves five different evaluations of the power function (three when $p = 1$ or $q = 1$) and represents about 200 basic floating point operations.

To adapt this algorithm to ratioquadrics, we have to replace f_p by g_p in Equation 11. This time, the squared distance ratio d^2 is the solution of a cubic equation (quadratic when $p = 1$ or $q = 1$) :

$$d^6 + A\ d^4 + B\ d^2 + C = 0 \tag{14}$$

Moreover, the star-shape property of the ratiospheroid guarantees that there is only one real root in Equation 14. This kind of cubic equations can be solved much more efficiently than the general three root case (about 50 basic floating point operations). It means that ratioquadrics are also a less expensive alternative to superquadrics when used in the blob model. **??** shows an example of a blobby object obtained by blending together 9 radioquadrics sources with various shape parameters.

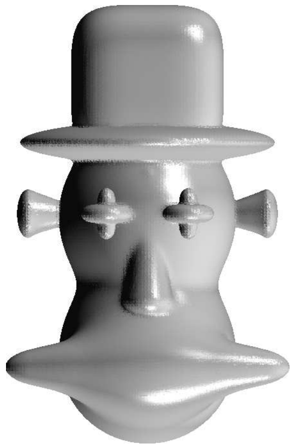


Figure 12 : *Blobby object with ratioquadric sources*

5 Extensions

Once the general principle of replacing function $f_p(t)$ by another one has been established, it becomes quite natural to try other functions than $g_p(t)$. The reader is warmly invited to experiment with his own functions. We have exhibit several conditions that appear desirable (but not imperative) for a candidate function $f_p(t)$ to provide interesting results :

- Bijective mapping : $\forall t \in [-1, 1] \quad \exists!(u, v) \in [-1, 1]^2 \quad f_p(t) = v \quad f_p(u) = t$
- Symmetric mapping : $\forall t \in [0, 1] \quad f_p(-t) = -f_p(t)$
- Reversibility (true or approximated) : $\forall t \in [-1, 1] \quad f_{1/p} \circ f_p(t) \simeq t$
- Separability (true or approximated) : $\forall (u, v) \in [-1, 1]^2 \quad f_p(uv) \simeq f_p(u) f_p(v)$

5.1 Quadratic Ratioconics and Ratioquadrics

A straightforward possibility is to raise the degree of the rational polynomial to get additional degrees of freedom to fulfill other mathematical conditions. For instance, the following quadratic rational function is a simple extension of Equation 6 which provides C^2/G^2continuity for the resulting curves and surfaces :

$$\forall t \in [-1,1] \quad h_p(t) = \mathrm{sign}(t)\ \frac{p|t| + |t|^2}{p^2 + p|t| + (1-p^2)|t|^2} \tag{15}$$

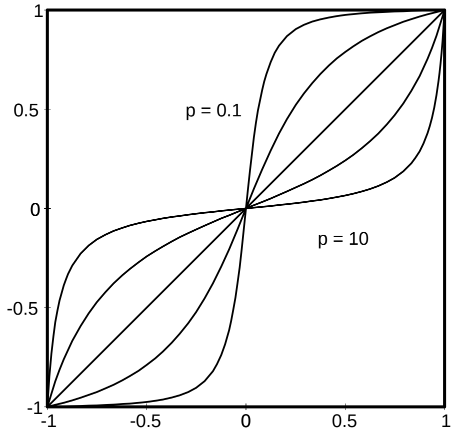


Figure 13 : *$h_p(t)$ for $p = 0.1, 0.4, 1, 2.5, 10$*

5.2 Ratiocrosses

If we try to increase the steepness of the function in the neighbourhood of $t = 0$, we obtain another interesting quadratic rational polynomial that can be used to generate different kinds of crosses :

$$\forall t \in [-1,1] \quad k_p(t) = \mathrm{sign}(t)\ \frac{p^2q|t| + p^2|t|^2 + (1-p)(p+pq+2q)|t|^2}{pq + p^2|t| + (1-p)(p+2q)|t|^2} \tag{16}$$

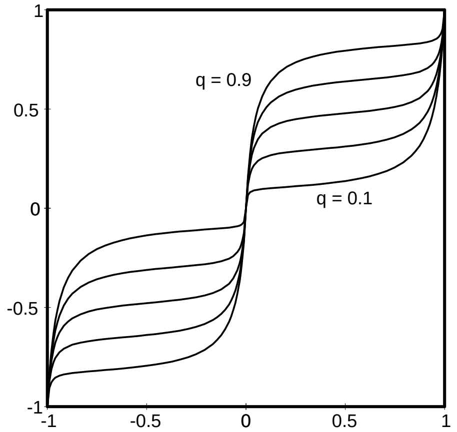


Figure 14 : *$k_p(t)$ for $p = 20$ and $q = 0.1, 0.3, 0.5, 0.7, 0.9$*

Two degrees of freedom are provided by Equation 16 : parameter $p \in [1, \infty)$ controls the roundness of the crosses (the lower the value of p, the rounder the corners) and parameter $q \in [0, 1]$ controls the width of the branches (the lower the value of q, the smaller the branches). The role of these two shape parameters is illustrated in Figure 15 and Figure 16. Two particular cases should be noted :

- A perfect roman cross is obtained when $p = \infty$. In that case q represents the ratio between the branche width and the global cross width.

- Linear ratiocircles, that can be considered as zero-width ratio crosses, are obtained when $q = 0$.

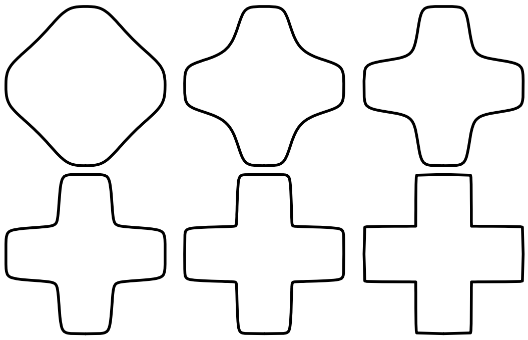


Figure 15 : *Ratiocrosses (from upper-left corner to lower-right corner,* $p = 2, 5, 10, 25, 100, 1000$ *and* $q = 0.35$*)*

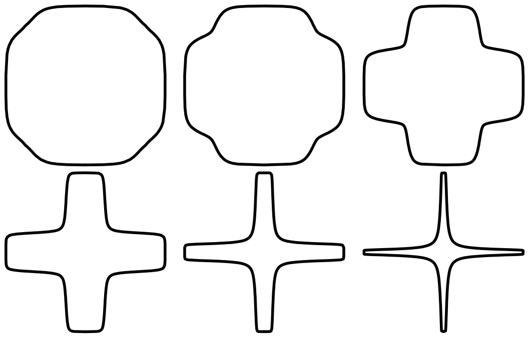


Figure 16 : *Ratiocrosses (from upper-left corner to lower-right corner,* $q = 0.8, 0.7, 0.5, 0.3, 0.1, 0.02$ *and* $p = 20$*)*

5.3 Multiparameter Ratioconics and Ratioquadrics

A last extension[2] of the ratioconics/ratioquadrics model can be mentionned. It consists in using different shape parameters for different parts of the same curve or surface. For instance, in the equation of the supercircle (Equation 2) one can use a shape parameter p that is no longer a constant but rather a function of parameter u. As far as a C^1 function is employed, the overall continuity of the curve will not be changed but the variety of the resulting shapes will be greatly extended (see Figure 17 for some examples). One can imagine a similar extension for implicit ratioconics (respectively ratioquadrics) by making the shape parameters depend on the polar (respectively spherical) coordinates of the points.

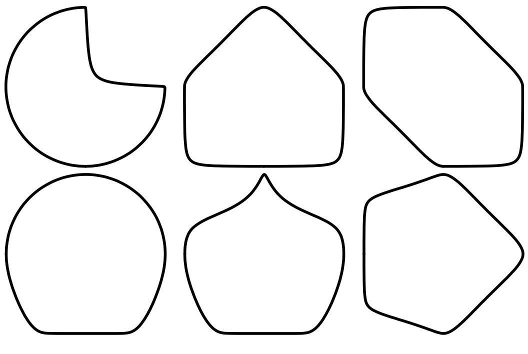


Figure 17 : *Multiparameter Ratiocircles*

[2]Note that this extension can also be applied to the superconics/superquadrics model.

6 Conclusion

In this paper, we have presented alternative models for superconics and superquadrics. These alternatives, called ratioconics and ratioquadrics, have been obtained by analyzing the key properties of the original models (i.e. bijective mapping, symmetric mapping, reversibility and separability of the signed power function) and finding a rational polynomial function that fulfills the same properties. The resulting curves and surfaces invole lower computation cost, provides better numerical robustness and offer higher orders of continuity. Finally, the primitives can also be easily plugged in the blob model to generate complex shapes based on blended ratioquadrics.

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