

## UE: APPLIED ALGORITHMS - 4TIN915U

(test – duration 1h30)

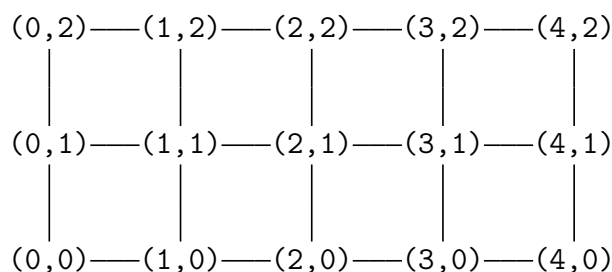
- *Answers can be done in English or in French* –
  - *Documents (course and personal notes) allowed* –  
*(the official lecture notes will be available)*
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### Tetris

A *tetris-graph* is a 2-connected subgraph of a 2D grid containing 3 or 4 squares. Recall that a  $k$ -connected graph is a connected graph that needs the removal of at least  $k$  vertices to be disconnected. Two cycles sharing a single vertex is a connected graph but not 2-connected. However, the  $5 \times 3$ -grid drawn below is a 2-connected graph.

**Question 1.** *Draw all the possible tetris-graphs with a dominating set of minimum size for each of them. We do not ask to prove optimality, just to draw an optimal solution.*

**Question 2.** *Give a tree-decomposition for the  $5 \times 3$ -grid as drawn below with the smallest possible width. Give the tree, the bags and their contents. [Hint: Check that all the properties hold. Try a tree-decomposition that is a path.]*

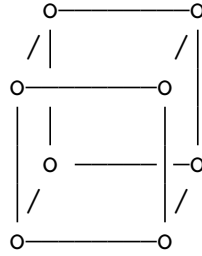


**Question 3.** *Conclude that the tree-width of tetris-graphs is no more than 3.*

### Covering Points by Spheres

Consider a set  $P$  of points in the 3D Euclidean space, that is  $P \subset \mathbb{R}^3$ . The *circularity* of  $P$  is the largest number  $c$  such that there exists a plan  $H \subset \mathbb{R}^3$  with a circle  $C \subset H$  containing  $c$  points of  $P$ .

For instance, if  $P$  is the set of the 8 points corresponding to the vertices of a cube (see the figure below), then the circularity of  $P$  is 4. Indeed, (1) each face defines a plan  $H$  with a circle  $C$  containing 4 vertices (there are other plans than these 6 faces containing 4 vertices as well); and (2) no other plan of  $\mathbb{R}^3$  can contain a circle with more than 4 vertices of the cube.



We will assume to be given two routines running in constant time:

- $\text{circle}(u, v, w)$  that, given 3 points  $u, v, w \in \mathbb{R}^3$ , returns a circle  $C$  containing  $u, v, w$ .
- $\text{check}(p, C)$  that says whether a point  $p$  belongs to a circle  $C$  or not.

In the case where  $u, v, w$  are aligned, we can assume that  $\text{circle}(u, v, w)$  returns a special circle  $C$  with infinite radius (in fact a line), and that  $\text{check}(p, C)$ , in this case, says whether  $p$  is aligned with  $u, v, w$  or not.

**Question 4.** Give an algorithm that computes the circularity of any point set  $P \subset \mathbb{R}^3$ . Describe its principle and give its complexity as a function of  $n = |P|$ .

A *sphere cover* for the point set  $P \subset \mathbb{R}^3$  is a collection  $\mathcal{C}$  of spheres such that every point of  $P$  belongs to the *surface* of some sphere of  $\mathcal{C}$ .

We consider the decision problem SPHERE COVER defined as follows:

**Instance:** A point set  $P \subset \mathbb{R}^3$  and  $k \in \mathbb{N}$ .

**Question:** Does  $P$  has a sphere cover of size  $k$ ?

We will also consider the optimization variant, called MINIMUM SPHERE COVER, that asks to find the minimum  $k$  such that  $P$  has a sphere cover of size  $k$ .

Recall that it takes a constant time to check whether a points  $p \in \mathbb{R}^3$  belongs to the surface of some sphere described by its center  $u$  and radius  $r$ . Indeed, it is sufficient to check that the Euclidean distance between  $u$  and  $p$  equals  $r$ .

**Question 5.** Give a reduction rule showing that SPHERE COVER admits a kernel whenever the problem is parametrized by  $k$  and the circularity  $c$  of  $P$ . (A complete justification is required here! You have to show that your reduction is valid and is polynomial (give its

complexity), and you have to express the size of the kernel as function of the parameters.)  
[Hint: Observe that the circularity of any subset of  $P$  is at most  $c$ .]

**Question 6.** Explain to which general problem seen in the course MINIMUM SPHERE COVER is a particular case. In particular, what are the sets  $B$ ,  $\mathcal{F}$ , and their size  $|B|$ ,  $|\mathcal{F}|$ ?

**Question 7.** Explain how to approximate MINIMUM SPHERE COVER in polynomial time. What is the approximation factor? Try to be accurate.

## Hitting Set

In this exercise, we consider again points in  $\mathbb{R}^3$ . We denote by  $G_n$  all the points corresponding to the vertices of a regular 3D grid  $n \times n \times n$ . Note that  $G_n$  has  $n^3$  points, and that  $G_2$  is the point set of the cube as depicted in the previous exercise.

We assume that each point  $u \in G_n$  is associated with a ball  $B_u$  centered at  $u$  and with some real radius  $r_u > 0$ . Depending on  $r_u$ , each ball  $B_u$  contains some subset of points of  $G_n$ . For instance, in the cube example, assuming unit length edges and  $r_u = 1$  for all  $u \in G_2$ , each ball  $B_u$  contains exactly 4 points of  $G_2$ : the center  $u$  and its 3 neighbors.

Let  $\mathcal{B}_n = \{B_u : u \in G_n\}$  be the collection of balls for the points of  $G_n$ . We are looking for a *hitting set* for  $\mathcal{B}_n$  that is in  $G_n$ , that is a subset  $S \subseteq G_n$  of minimum size that intersects each ball of  $G_n$ . More precisely, for each ball  $B_u \in \mathcal{B}_n$  there must exist a point  $s \in S \cap B_u$ .

**Question 8.** Give a hitting set  $S$  of minimum size for  $\mathcal{B}_2$  for the cube with unit length edges and unit radii.

**Question 9.** In the light of Question 7, explain how to approximate the MINIMUM HITTING SET problem for  $\mathcal{B}_n$  in polynomial time, and give the approximation factor. Try to be accurate. [Hint: The size of  $\mathcal{B}_n$  is not  $n$ .]

**FIN.**