

CATALAN PATHS AND QUASI-SYMMETRIC FUNCTIONS

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ABSTRACT. We investigate the quotient ring R of the ring of formal power series $\mathbb{Q}[[x_1, x_2, \dots]]$ over the closure of the ideal generated by non-constant quasi-symmetric functions. We show that a Hilbert basis of the quotient is naturally indexed by Catalan paths (infinite Dyck paths). We also give a filtration of ideals related to Catalan paths from $(0, 0)$ and above the line $y = x - k$. We investigate as well the quotient ring R_n of polynomial ring in n variables over the ideal generated by non-constant quasi-symmetric polynomials. We show that the dimension of R_n is bounded above by the n th Catalan number.

1. INTRODUCTION

The ring $Qsym$ of quasi-symmetric functions was introduced by Gessel [11] as a source of generating functions for P -partitions [16]. Since then, quasi-symmetric functions have appeared in many combinatorial contexts [6, 16, 17]. The relation of $Qsym$ to the ring of symmetric functions was first clarified by Malvenuto and Reutenauer [15] via a graded Hopf duality to the Solomon descent algebras, then Gelfand *et. al.* [10] defined the graded Hopf algebra NC of non-commutative symmetric functions and identified it with the Solomon descent algebra. In recent literature, we see a growing interest in quasi-symmetric functions and non-commutative symmetric functions as refinements of the ring of symmetric functions.

One unexplored avenue is as an analogue of the (symmetric) harmonic spaces. A classic combined result of Artin and Steinberg [1, 18] shows that the quotient ring of the polynomial ring $\mathbb{Q}[x_1, x_2, \dots, x_n]$ in n variables over the ideal \mathcal{I}_n generated by non-constant symmetric polynomials has dimension $n!$. In fact, this space is a graded symmetric group module that affords the left regular representation. Refinement and generalization of this result has lead to an explosion of incredible results and conjectures, for example see [4, 5, 7, 9, 12] for a small portion of this.

Here we are interested in the quotient ring R of the ring of formal power series $\mathbb{Q}[[x_1, x_2, x_3, \dots]]$ over the closure of the homogeneous ideal \mathcal{J} generated by all non-constant quasi-symmetric functions. That is the quotient $R = \mathbb{Q}[[x_1, x_2, x_3, \dots]] / \overline{\mathcal{J}}$.

To every monomial $x_1^{\tilde{\alpha}_1} x_2^{\tilde{\alpha}_2} x_3^{\tilde{\alpha}_3} \dots$ (of finite total degree) in $\mathbb{Q}[[x_1, x_2, x_3, \dots]]$ we associate a path in the plane as follows: $(0, 0) \rightarrow (\tilde{\alpha}_1, 0) \rightarrow (\tilde{\alpha}_1, 1) \rightarrow (\tilde{\alpha}_1 + \tilde{\alpha}_2, 1) \rightarrow (\tilde{\alpha}_1 + \tilde{\alpha}_2, 2) \rightarrow (\tilde{\alpha}_1 + \tilde{\alpha}_2 + \tilde{\alpha}_3, 2) \rightarrow \dots$. If this path remains above the line $y = x$ we say that the path is a *Catalan path* (or infinite Dyck path). Our main result is

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Theorem 1.1. *A monomial Hilbert basis of R is given by the monomials of $\mathbb{Q}[[x_1, x_2, x_3, \dots]]$ corresponding to Catalan paths.*

We also consider a special filtration of ideals $\mathcal{J}^{(e)}$ and their respective quotients, such that $\mathcal{J} = \mathcal{J}^{(0)}$ and $\mathcal{J}^{(e)} \subseteq \mathcal{J}^{(e+1)}$. The Hilbert basis of each quotient is indexed by paths above the line $y = x - e$.

Let R_n denote the quotient of the polynomial ring $\mathbb{Q}[x_1, x_2, \dots, x_n]$ over the ideal \mathcal{J}_n generated by all non-constant quasi-symmetric polynomials. Recall that $C_n = \frac{1}{n+1} \binom{2n}{n}$ are the famous Catalan numbers. The passage from infinitely many variables to finitely many variables is a priori non-trivial, and requires more work.

Theorem 1.2. $\dim R_n \leq C_n$.

In fact we expect equality to hold in Theorem 1.2 and we are still in the process of completing a proof of this together with François Bergeron and Adriano Garsia, who have discovered the spaces R_n completely independently and in the same period that we did. They have conjectured many of the results presented here, and much more. The results for R_n are extremely interesting, and are the object of an ongoing collaboration with F. Bergeron and A. Garsia. We plan to write at least two papers: one [2] dedicated to a proof of equality in Theorem 1.2, and another [3] to investigate further properties of R_n and its generalization. In particular, the finite version of the successive quotients by the ideals $\mathcal{J}_n^{(e)}$ is related to the work of [5]. Much of these results can be explained in a more general framework and will be the object of further study. We are convinced that these results are but the tip of a new iceberg. In particular, we would like to find any natural algebras acting on these spaces. What are the possible generalizations and specializations of the super-harmonics?

We underline here that F. Hivert [13] has developed an action of the Hecke algebra for which a polynomial is invariant if and only if it is quasi-symmetric. Unfortunately Hivert's action is not compatible with multiplication and does not preserve the ideal \mathcal{J}_n , hence it does not induce the desired action on the quotient. It is still interesting to note that Hivert's action is also related to Catalan numbers.

In Section 2 we recall appropriate definitions. In Section 3 we introduce a special family of generators for the ideal \mathcal{J} and the associated filtration $\mathcal{J}^{(e)}$. In Section 4 we use these generators to show that the monomials corresponding to Catalan paths span our quotient, as well as the analogous result for $\mathcal{J}^{(e)}$. Theorem 1.2 follows from this section. To complete the proof of Theorem 1.1, we use a Gröbner basis argument in Section 5 to show independence.

2. BASIC DEFINITIONS

A composition $\alpha = [\alpha_1, \alpha_2, \dots, \alpha_k]$ of a positive integer d is an ordered list of positive integers whose sum is d . We denote this by $\alpha \models d$. We call the integers α_i the *parts* of α , and denote the number of parts in α by $\ell(\alpha)$. Given two compositions $\alpha = [\alpha_1, \alpha_2, \dots, \alpha_k]$ and $\beta = [\beta_1, \beta_2, \dots, \beta_\ell]$, we denote by $\alpha\beta$ the concatenation product $[\alpha_1, \alpha_2, \dots, \alpha_k, \beta_1, \beta_2, \dots, \beta_\ell]$. Also, there exists a natural one-to-one correspondence between compositions of d and subsets of $\{1, 2, \dots, d-1\}$. If $A = \{a_1, a_2, \dots, a_{k-1}\} \subset [d-1]$, where $a_1 < a_2 < \dots < a_{k-1}$, then A corresponds to the composition, $\alpha = [a_1 - a_0, a_2 - a_1, \dots, a_k - a_{k-1}]$, where $a_0 = 0$ and $a_k = d$. For ease of notation, we shall denote the set corresponding to a given composition α by $D(\alpha)$. For compositions α and β we say that α is a *refinement* of β if $D(\beta) \subset D(\alpha)$, and denote this by $\alpha \preceq \beta$.

For any composition $\alpha = [\alpha_1, \alpha_2, \dots, \alpha_k]$ of d we denote by M_α the *monomial quasi-symmetric function* [11]: $M_\alpha(x_1, x_2, \dots) = \sum_{i_1 < i_2 < \dots < i_k} x_{i_1}^{\alpha_1} \dots x_{i_k}^{\alpha_k}$. This is a homogeneous infinite series of degree d . We define $M_0 = 1$, where 0 denotes the unique empty composition of 0. It is known from the work of Gessel that the monomial quasi-symmetric functions form a linear basis of a ring (in fact a Hopf algebra) $Qsym$ of quasi-symmetric functions.

An other useful basis of the ring $Qsym_n$ is given by the *fundamental quasi-symmetric function* [11]:

$$F_\alpha(x_1, x_2, \dots) = \sum_{\alpha \triangleright \beta} M_\beta(x_1, x_2, \dots) = \sum_{\substack{j_1 \leq j_2 \leq \dots \leq j_d \\ i \in D(\alpha) \Rightarrow j_i < j_{i+1}}} x_{j_1} x_{j_2} \dots x_{j_d}.$$

Fundamental quasi-symmetric functions satisfy the following obvious, but crucial, relations. For $\alpha = [\alpha_1, \alpha_2, \dots, \alpha_k] \models d$,

(2.1)

$$F_\alpha(x_1, x_2, \dots) = \begin{cases} x_1 F_{[\alpha_1-1, \alpha_2, \dots, \alpha_k]}(x_1, x_2, \dots) + F_\alpha(x_2, x_3, \dots) & \text{if } \alpha_1 > 1, \\ x_1 F_{[\alpha_2, \alpha_3, \dots, \alpha_k]}(x_2, x_3, \dots) + F_\alpha(x_2, x_3, \dots) & \text{if } \alpha_1 = 1. \end{cases}$$

Here $F_\alpha(x_2, x_3, \dots)$ is the function $F_\alpha(x_1, x_2, \dots)$ in which the variable x_i is replaced by x_{i+1} . These relations are the key ingredients in our proof.

In the following we have to consider generalized (infinite) compositions. That is a sequence $\tilde{\alpha} = (\tilde{\alpha}_1, \tilde{\alpha}_2, \dots)$ such that the parts $\tilde{\alpha}_j \geq 0$ for $j \geq 1$ (we allow some parts to be zero) and the sum of the parts $d(\tilde{\alpha}) = \sum \tilde{\alpha}_i < \infty$. We say that $\tilde{\alpha}$ is a generalized composition of $d(\tilde{\alpha}) < \infty$. We use a “ $\tilde{}$ ” to indicate that we have a generalized composition and no “ $\tilde{}$ ” if the composition is *standard*, that is without zeros. We also consider generalized compositions of finite length and denote by $\ell(\tilde{\alpha})$ the number of parts of $\tilde{\alpha}$. The concatenation of a finite length generalized composition $\tilde{\alpha}$ with an infinite one $\tilde{\beta}$ is denoted by $\tilde{\alpha}\tilde{\beta}$. We also write $\tilde{\alpha} + \tilde{\beta}$ and $\tilde{\alpha} \leq \tilde{\beta}$ to denote the componentwise sum and componentwise inequalities, respectively. For an infinite generalized composition $\tilde{\alpha}$, since $d(\tilde{\alpha}) < \infty$, only finitely many parts of $\tilde{\alpha}$ are non-zero. Thus there is always a finite generalized composition $\tilde{\nu}$ such that $\tilde{\alpha} = \tilde{\nu}00\dots$.

In this paper, we devote our attention to the ideal $\mathcal{J} = \langle F_\alpha(x_1, x_2, \dots) \rangle_{\alpha \models d > 0}$ of $\mathbb{Q}[[x_1, x_2, \dots]]$ generated by the non-constant quasi-symmetric functions, and consider the quotient

$$(2.2) \quad R = \mathbb{Q}[[x_1, x_2, \dots]] / \overline{\mathcal{J}},$$

where $\overline{\mathcal{J}}$ denotes the closure (with respect to the standard topology with formal power series) of \mathcal{J} in $\mathbb{Q}[[x_1, x_2, \dots]]$.

3. THE GENERATORS $G_{\tilde{\alpha}}$

We exploit the relation (2.1) to construct a set $\{G_{\tilde{\alpha}}\} \subseteq \mathcal{J}$ indexed by the generalized (infinite) composition $\tilde{\alpha}$ such that there exists a factorization $\tilde{\alpha} = \tilde{\pi}\tilde{\rho}$ where $d(\tilde{\pi}) - \ell(\tilde{\pi}) \geq 0$. We first define recursively the functions $G_{\tilde{\alpha}}$ for all infinite generalized composition $\tilde{\alpha}$. Then in Lemma 3.1 we characterize the $\tilde{\alpha}$ obtained from the transitive closure of the $G_{\tilde{\alpha}} \in \mathcal{J}$. Let $\tilde{\alpha} = \tilde{\nu}00\dots$ where $\ell(\tilde{\nu}) < \infty$ and the last part of $\tilde{\nu}$ is non-zero, or $\tilde{\alpha} = 00\dots$. Our definition is recursive on $n = \ell(\tilde{\nu})$. If

$\tilde{\nu} = \nu$ is a standard composition, then let

$$(3.1) \quad G_{\tilde{\alpha}} = F_{\nu}(x_1, x_2, \dots).$$

If $\ell(\tilde{\nu}) = 0$, then this formula gives $G_{00\dots} = 1$. Assume now that $\tilde{\nu}$ is non-standard and let $\tilde{\nu} = \tilde{\gamma}0a\beta$ be the unique factorization of $\tilde{\nu}$ such that $a > 0$ is a positive integer, β is a (possibly empty) standard composition and $\tilde{\gamma}$ is a (possibly empty) generalized composition. For $\tilde{\alpha} = \tilde{\gamma}0a\beta00\dots$ and $k = \ell(\tilde{\gamma}0) = \ell(\tilde{\gamma}) + 1$, we define

$$(3.2) \quad G_{\tilde{\alpha}} = G_{\tilde{\gamma}a\beta0\dots} - x_k G_{\tilde{\gamma}(a-1)\beta0\dots}$$

Both terms on the right are well defined by induction since $\ell(\tilde{\gamma}a\beta) = \ell(\tilde{\nu}) - 1 < n$.

We now characterize the transitive closure of the definition (3.1) and (3.2) within \mathcal{J} . At this point it is useful to introduce the following family of ideals. For any $e \geq 0$, let $\mathcal{J}^{(e)} = \langle F_{\alpha} : \exists \pi\rho = \alpha, d(\pi) - \ell(\pi) \geq e \rangle$. This is a filtration $\mathcal{J}^{(e)} \subseteq \mathcal{J}^{(e+1)}$ such that $\mathcal{J} = \mathcal{J}^{(0)}$. For a generalized composition $\tilde{\alpha}$, we say that it *reaches level* e if there exists a factorization $\tilde{\alpha} = \tilde{\pi}\tilde{\rho}$ such that $d(\tilde{\pi}) - \ell(\tilde{\pi}) \geq e$.

Lemma 3.1.

1. If $\tilde{\alpha}$ reaches level e , then $G_{\tilde{\alpha}} \in \mathcal{J}^{(e)}$.
2. Conversely, in (3.2), if $\tilde{\gamma}(a-1)\beta0\dots$ reaches level e , then $\tilde{\alpha}$ reaches level e .

Proof. For the first statement we proceed by induction on $\ell(\tilde{\nu})$ where $\tilde{\alpha} = \tilde{\nu}00\dots$ and the last part of $\tilde{\nu}$ is non-zero. If $\ell(\tilde{\nu}) = 0$, then $G_{00\dots} = 1$ is not in any of the ideals $\mathcal{J}^{(e)}$. Assume that $\ell(\tilde{\nu}) > 0$. We first consider the case when $\tilde{\nu} = \nu$ is a standard composition. If $\tilde{\alpha}$ reaches level e , then so is ν and we have $G_{\tilde{\alpha}} = F_{\nu}(x_1, x_2, \dots) \in \mathcal{J}^{(e)}$. If $\tilde{\nu}$ is a non-standard generalized composition, then let $\tilde{\alpha} = \tilde{\pi}\tilde{\rho}$ be the factorization such that $d(\tilde{\pi}) - \ell(\tilde{\pi}) \geq e$, and let $\tilde{\alpha} = \tilde{\nu}00\dots = \tilde{\gamma}0a\beta00\dots$ be the factorization used in (3.2). If $\tilde{\pi}$ is an initial factor of $\tilde{\gamma}$, then it is clearly an initial factor of both $\tilde{\gamma}a\beta00\dots$ and $\tilde{\gamma}(a-1)\beta00\dots$ and they both reach level e . By the induction hypothesis, both $G_{\tilde{\gamma}a\beta0\dots}$ and $G_{\tilde{\gamma}(a-1)\beta0\dots}$ are in $\mathcal{J}^{(e)}$ and in turn $G_{\tilde{\alpha}} \in \mathcal{J}^{(e)}$. If we now assume that $\tilde{\pi} = \tilde{\gamma}0$, then $d(\tilde{\gamma}) - \ell(\tilde{\gamma}) = d(\tilde{\pi}) - (\ell(\tilde{\pi}) - 1) \geq e + 1 > e$ and again the induction hypothesis can be applied to (3.2) to show that $G_{\tilde{\alpha}} \in \mathcal{J}^{(e)}$.

We are left to check the case where $\tilde{\pi} = \tilde{\gamma}0a\tilde{\mu}$. For the first term in (3.2), $\tilde{\gamma}a\tilde{\mu}$ is an initial factor and $d(\tilde{\mu}a\tilde{\gamma}) - \ell(\tilde{\gamma}a\tilde{\mu}) = d(\tilde{\pi}) - (\ell(\tilde{\pi}) - 1) \geq e + 1 > e$. The induction hypothesis gives that $G_{\tilde{\gamma}a\beta0\dots} \in \mathcal{J}^{(e)}$. For the second term indexed by $\tilde{\gamma}(a-1)\beta00\dots$ we have $d(\tilde{\gamma}(a-1)\tilde{\mu}) - \ell(\tilde{\gamma}(a-1)\tilde{\mu}) = (d(\tilde{\pi}) - 1) - (\ell(\tilde{\pi}) - 1) \geq e$. Again the induction hypothesis gives us that $G_{\tilde{\gamma}(a-1)\beta0\dots} \in \mathcal{J}^{(e)}$, concluding the proof that $G_{\tilde{\alpha}} \in \mathcal{J}^{(e)}$.

For the second statement of the lemma let $\tilde{\gamma}(a-1)\beta00\dots = \tilde{\pi}\tilde{\rho}$ be a factorization such that $d(\tilde{\pi}) - \ell(\tilde{\pi}) \geq e$. If $\tilde{\pi}$ is an initial factor of $\tilde{\gamma}$ then it is clear that $\tilde{\alpha}$ reaches level e . On the other hand if $\tilde{\pi} = \tilde{\gamma}(a-1)\tilde{\mu}$, then we have $d(\tilde{\gamma}0a\tilde{\mu}) - \ell(\tilde{\gamma}0a\tilde{\mu}) = (d(\tilde{\gamma}(a-1)\tilde{\mu}) + 1) - (\ell(\tilde{\gamma}(a-1)\tilde{\mu}) + 1) \geq e$. Thus $\tilde{\alpha}$ reaches level e . \square

In light of the previous lemma, let $\mathcal{G}^{(e)}$ denote the set of all generalized infinite compositions $\tilde{\alpha}$ reaching level e , that satisfy $d(\tilde{\pi}) - \ell(\tilde{\pi}) \geq e$. We remark that the set $\{G_{\tilde{\alpha}}\}_{\tilde{\alpha} \in \mathcal{G}^{(e)}}$ constructed above is contained in $\mathcal{J}^{(e)}$ and contains $\{F_{\alpha}(x_1, x_2, \dots) : \exists \alpha = \pi\rho, d(\pi) - \ell(\pi) \geq e\}$. Hence we have

Lemma 3.2. $\mathcal{J}^{(e)} = \langle G_{\tilde{\alpha}} \rangle_{\tilde{\alpha} \in \mathcal{G}^{(e)}}$. \square

Our next task is to characterize the leading monomial of each function $G_{\tilde{\alpha}}$. Before this we need to specify which monomial order we use. Let $X^{\tilde{\alpha}} = x_1^{\tilde{\alpha}_1} x_2^{\tilde{\alpha}_2} \cdots$ and $X^{\tilde{\beta}} = x_1^{\tilde{\beta}_1} x_2^{\tilde{\beta}_2} \cdots$ be any two monomials where $\tilde{\alpha}$ and $\tilde{\beta}$ are two generalized infinite compositions. We say that $X^{\tilde{\alpha}} \leq_{lex} X^{\tilde{\beta}}$ if and only if $d(\tilde{\alpha}) > d(\tilde{\beta})$, or $d(\tilde{\alpha}) = d(\tilde{\beta})$ and the leftmost non-zero entry in $[\tilde{\beta}_1 - \tilde{\alpha}_1, \tilde{\beta}_2 - \tilde{\alpha}_2, \dots]$ is positive. The order \leq_{lex} is a classical monomial order in the sense that it is a total order and if $X^{\tilde{\alpha}} \leq_{lex} X^{\tilde{\beta}}$, then $X^{\tilde{\alpha}} X^{\tilde{\gamma}} = X^{\tilde{\alpha} + \tilde{\gamma}} \leq_{lex} X^{\tilde{\beta} + \tilde{\gamma}} = X^{\tilde{\beta}} X^{\tilde{\gamma}}$. Here the sum of generalized compositions is componentwise.

For any formal power series $P = P(x_1, x_2, \dots) \in \mathbb{Q}[[x_1, x_2, \dots]]$ we let $LM(P)$ denote the leading monomial of P . That is $LM(P)$ is the monomial of P with non-zero coefficient of smallest degree and largest in lexicographic order. In other words, the leading monomial for the order \leq_{lex} . We let $LC(P)$ denote the coefficient of $LM(P)$ in P . Remark that for any two functions P and Q , we have $LM(PQ) = LM(P)LM(Q)$. We need the following result which is the extension for the G -functions of Relation (2.1) for the F -functions.

Lemma 3.3. *Let $\tilde{\alpha} = b \tilde{\rho}$ be any generalized infinite composition and $b \geq 0$.*

$$(3.3) \quad G_{\tilde{\alpha}}(x_1, x_2, \dots) = \begin{cases} G_{\tilde{\rho}}(x_2, x_3, \dots) & \text{if } b = 0, \\ x_1 G_{(b-1)\tilde{\rho}} + R_{\tilde{\alpha}}(x_2, x_3, \dots) & \text{if } b > 0. \end{cases}$$

Proof. We remark that the function $G_{\tilde{\rho}}(x_2, x_3, \dots)$ is the function $G_{\tilde{\rho}}(x_1, x_2, \dots)$ in which the variable x_i is replaced by x_{i+1} . We let $\tilde{\alpha} = \tilde{\nu} 0 0 \cdots$ and proceed by induction on $\ell(\tilde{\nu})$. If $\ell(\tilde{\nu}) = 0$ then we have $G_{0 0 \dots} = 1$ and (3.3) is valid. In the following we extensively use the relation (2.1). If $\tilde{\alpha} = b \rho 0 0 \cdots$, then Equation (3.1) gives us $G_{\tilde{\alpha}} = F_{b \rho}(x_1, x_2, \dots)$. If $b > 1$, then we get

$$\begin{aligned} G_{b \rho 0 \dots} &= F_{b \rho}(x_1, x_2, \dots) = x_1 F_{(b-1)\rho}(x_1, x_2, \dots) + F_{b \rho}(x_2, x_3, \dots) \\ &= x_1 G_{(b-1)\rho 0 \dots} + R_{b \rho 0 \dots}(x_2, x_3, \dots), \end{aligned}$$

and (3.3) follows for this case with $R_{\tilde{\alpha}}(x_2, x_3, \dots) = F_{b \rho}(x_2, x_3, \dots)$.

For $b = 1$, we first need to understand (3.3) in the case $G_{0 \rho 0 \dots}$. For this assume that $\rho = a \beta$. If $a > 1$, then the Definitions (3.1) and (3.2) give

$$\begin{aligned} G_{0 a \beta 0 \dots} &= G_{a \beta 0 \dots} - x_1 G_{(a-1)\beta 0 \dots} = (F_{a \beta} - x_1 F_{(a-1)\beta})(x_1, x_2, \dots) \\ &= F_{a \beta}(x_2, x_3, \dots) = G_{a \beta 0 \dots}(x_2, x_3, \dots) \end{aligned}$$

If $a = 1$, then we use the induction hypothesis on $\ell(0 \beta) = \ell(0 1 \beta) - 1 < \ell(\tilde{\nu})$ to get

$$\begin{aligned} G_{0 1 \beta 0 \dots} &= G_{1 \beta 0 \dots} - x_1 G_{0 \beta 0 \dots} = F_{1 \beta}(x_1, x_2, \dots) - x_1 F_{\beta}(x_2, x_3, \dots) \\ &= F_{1 \beta}(x_2, x_3, \dots) = G_{1 \beta 0 \dots}(x_2, x_3, \dots) \end{aligned}$$

Now we can go back to the case of $\tilde{\alpha} = 1 \rho 0 \cdots$:

$$\begin{aligned} G_{1 \rho 0 \dots} &= F_{1 \rho}(x_1, x_2, \dots) = x_1 F_{\rho}(x_2, x_3, \dots) + F_{1 \rho}(x_2, x_3, \dots) \\ &= x_1 G_{0 \rho 0 \dots} + R_{1 \rho 0 \dots}(x_2, x_3, \dots), \end{aligned}$$

where $R_{1 \rho 0 \dots}(x_2, x_3, \dots) = F_{1 \rho}(x_2, x_3, \dots)$.

We then consider when $\tilde{\alpha} = \tilde{\gamma} 0 a \beta 0 0 \cdots$. This is the factorization needed to use (3.2) with $k = \ell(\tilde{\gamma}) + 1$. If $\tilde{\gamma}$ is empty, then we have $b = 0$ and we are in the case considered above. Assume that $\tilde{\gamma} = b \tilde{\mu}$. If $b = 0$, then applying the induction hypothesis we have

$$\begin{aligned} G_{0 \tilde{\mu} 0 a \beta 0 \dots}(x_1, x_2, \dots) &= G_{0 \tilde{\mu} a \beta 0 \dots}(x_1, x_2, \dots) - x_k G_{0 \tilde{\mu} (a-1) \beta 0 \dots}(x_1, x_2, \dots) \\ &= G_{\tilde{\mu} a \beta 0 \dots}(x_2, x_3, \dots) - x_{(k-1)+1} G_{\tilde{\mu} (a-1) \beta 0 \dots}(x_2, x_3, \dots) \\ &= G_{\tilde{\mu} 0 a \beta 0 \dots}(x_2, x_3, \dots). \end{aligned}$$

Here remark we that even though $\ell(\tilde{\mu}) + 1 = k - 1$, we have to replace x_{k-1} by $x_{(k-1)+1} = x_k$ in the recurrence for $G_{\tilde{\mu} 0 a \beta 0 \dots}(x_2, x_3, \dots)$. For $b > 0$,

$$\begin{aligned} G_{b \tilde{\mu} 0 a \beta 0 \dots} &= G_{b \tilde{\mu} a \beta 0 \dots} - x_k G_{b \tilde{\mu} (a-1) \beta 0 \dots} \\ &= x_1 G_{(b-1) \tilde{\mu} a \beta 0 \dots} + R_{b \tilde{\mu} a \beta 0 \dots}(x_2, x_3, \dots) \\ &\quad - x_k (x_1 G_{(b-1) \tilde{\mu} (a-1) \beta 0 \dots} + R_{b \tilde{\mu} (a-1) \beta 0 \dots}(x_2, x_3, \dots)) \\ &= x_1 (G_{(b-1) \tilde{\mu} a \beta 0 \dots} - x_k G_{(b-1) \tilde{\mu} (a-1) \beta 0 \dots}) \\ &\quad + R_{b \tilde{\mu} a \beta 0 \dots}(x_2, x_3, \dots) - x_k R_{b \tilde{\mu} (a-1) \beta 0 \dots}(x_2, x_3, \dots) \\ &= x_1 G_{(b-1) \tilde{\mu} 0 a \beta 0 \dots} + R_{b \tilde{\mu} 0 a \beta 0 \dots}(x_2, x_3, \dots), \end{aligned}$$

where $R_{b \tilde{\mu} 0 a \beta 0 \dots}(x_2, x_3, \dots) = (R_{b \tilde{\mu} a \beta 0 \dots} - x_k R_{b \tilde{\mu} (a-1) \beta 0 \dots})(x_2, x_3, \dots)$. \square

Corollary 3.4. *For any generalized composition $\tilde{\alpha}$ we have $LM(G_{\tilde{\alpha}}) = X^{\tilde{\alpha}}$.*

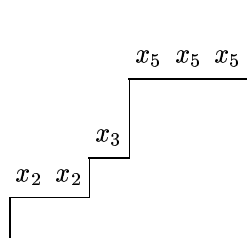
Proof. Let $\tilde{\alpha} = \tilde{\nu} 0 0 \dots$. We proceed by induction on $\ell(\tilde{\nu})$ and the degree $d = d(\tilde{\alpha}) = \sum \tilde{\alpha}_i$. If $\ell(\tilde{\nu}) = 0$ we have $G_{0 0 \dots} = 1 = X^{0 0 \dots}$. If $\ell(\tilde{\nu}) \geq 1$, then let $\tilde{\alpha} = b \tilde{\rho}$ as in Lemma 3.3. If $b = 0$, then the induction hypothesis on $\ell(\tilde{\nu})$ gives $LM(G_{0 \tilde{\rho}}) = LM(G_{\tilde{\rho}}(x_2, x_3, \dots)) = x_1^0 x_2^{\tilde{\rho}_1} x_3^{\tilde{\rho}_2} \dots = X^{\tilde{\alpha}}$. Now if $b > 0$ we use the second part of Lemma 3.3 and the induction hypothesis on d , and get $LM(x_1 G_{(b-1) \tilde{\mu}} + R_{\tilde{\alpha}}(x_2, x_3, \dots)) = x_1 LM(G_{(b-1) \tilde{\mu}}) = X^{\tilde{\alpha}}$. \square

Remark 3.5. From the above corollary, by triangularity, it is clear that the set $\{G_{\tilde{\alpha}}\}$ for all $\tilde{\alpha}$ forms a Hilbert basis of $\mathbb{Q}[[x_1, x_2, \dots]]$. We will see in Section 5 that in fact $\{G_{\tilde{\alpha}}\}_{\tilde{\alpha} \in \mathcal{G}^{(e)}}$ forms a Hilbert basis of $\mathcal{J}^{(e)}$.

4. IT IS AT MOST CATALAN.

Let $\mathcal{Q}^{(e)} = \{G_{\tilde{\alpha}}\}_{\tilde{\alpha} \in \mathcal{G}^{(e)}}$ be the generating set of $\mathcal{J}^{(e)}$ constructed in Lemma 3.2. In this section we show that after reduction, at most the monomials corresponding to Catalan paths form a Hilbert basis of $R = \mathbb{Q}[[x_1, x_2, \dots]]/\mathcal{J}$. For this we reduce every other monomial to these. In fact for $R^{(e)} = \mathbb{Q}[[x_1, x_2, \dots]]/\mathcal{J}^{(e)}$, we show that at most the monomials corresponding to paths above the line $y = x - e$ form a Hilbert basis of $R^{(e)}$, for all $e \geq 0$. We conclude this section with the corresponding result for finitely many variables, $R_n^{(e)} = \mathbb{Q}[x_1, x_2, \dots, x_n]/\mathcal{J}_n^{(e)}$, which is a generalization of Theorem 1.2.

Given any generalized infinite composition $\tilde{\alpha}$ we associate a unique path in the plane with steps going north or east. More precisely, for $\tilde{\alpha} = (\tilde{\alpha}_1, \tilde{\alpha}_2, \tilde{\alpha}_3, \dots)$, we construct the path $(0, 0) \rightarrow (\tilde{\alpha}_1, 0) \rightarrow (\tilde{\alpha}_1, 1) \rightarrow (\tilde{\alpha}_1 + \tilde{\alpha}_2, 1) \rightarrow (\tilde{\alpha}_1 + \tilde{\alpha}_2, 2) \rightarrow (\tilde{\alpha}_1 + \tilde{\alpha}_2 + \tilde{\alpha}_3, 2) \rightarrow \dots$. For example, for $\tilde{\alpha} = 0 2 1 0 3 0 0 \dots$ we have the path



For every east step at height $i - 1$ we associate a variable x_i . The product of all the variables associated to a path encoded by $\tilde{\alpha}$ is denoted $X^{\tilde{\alpha}}$. We now remark that for any factorization $\tilde{\alpha} = \tilde{\pi} \tilde{\rho}$, the rightmost coordinate of the path at height $\ell(\tilde{\pi}) - 1$ is $(d(\tilde{\pi}), \ell(\tilde{\pi}) - 1)$.

Definition 4.1. For an integer $e \geq 0$, we say that a generalized composition $\tilde{\alpha}$ is of type e -Catalan if its associated path remains above the line $y = x - e$. That is, every coordinate (x_i, y_i) of the path is such that $x_i - y_i \leq e$.

Lemma 4.2. *The monomials of $\mathbb{Q}[[x_1, x_2, \dots]]$ corresponding to paths remaining above the line $y = x - e$ contains a Hilbert basis of the quotient $R^{(e)}$.*

Proof. Let $X^{\tilde{\alpha}}$ be any monomial of degree d . If the path corresponding to $\tilde{\alpha}$ goes under the line $y = x - e$, then let $\tilde{\alpha} = \tilde{\pi}\tilde{\rho}$ be any factorization such that the coordinate $(d(\tilde{\pi}), \ell(\tilde{\pi}) - 1)$ is under the line $y = x - e$. That is $d(\tilde{\pi}) - \ell(\tilde{\pi}) \geq e$. From Lemma 3.1 we conclude that the function $G_{\tilde{\alpha}}$ with leading monomial $X^{\tilde{\alpha}}$ is in $\mathcal{J}^{(e)}$. This monomial can thus be replaced by monomials of degree d but strictly smaller with respect to $<_{lex}$. Repeating this step (possibly countably many times) with the next largest monomial going under the line $y = x - e$, any monomial $X^{\tilde{\alpha}}$ can be reduced modulo the ideal $\overline{\mathcal{J}^{(e)}}$ to a series containing only monomials $X^{\tilde{\beta}}$ where $\tilde{\beta}$ is of type e -Catalan. \square

We are now in position to generalize Theorem 1.2 and prove it. For this, note that the quasi-symmetric polynomials in n variables are defined by setting $0 = x_{n+1} = x_{n+2} = \dots$ in the quasi-symmetric functions. That is $F_{\alpha}(x_1, x_2, \dots, x_n) = F_{\alpha}(x_1, x_2, \dots, x_n, 0, 0, \dots)$. We then define $J_n^{(e)} = \langle F_{\alpha}(x_1, x_2, \dots, x_n) : \alpha \text{ reaches level } e \rangle$ and $R_n^{(e)} = \mathbb{Q}[x_1, x_2, \dots, x_n] / J_n^{(e)}$. Similarly we set $G_{\tilde{\alpha}}(x_1, x_2, \dots, x_n) = G_{\tilde{\alpha}}(x_1, x_2, \dots, x_n, 0, 0, \dots)$. It is clear that Lemma 3.1 holds for $J_n^{(e)}$ in the same way. More over if $\tilde{\alpha} = \tilde{\nu} 0 0 \dots$ for $\ell(\tilde{\nu}) = n$ then

$$LM(G_{\tilde{\alpha}}(x_1, x_2, \dots, x_n)) = LM(G_{\tilde{\alpha}}(x_1, x_2, \dots, x_n, 0, 0, \dots)) = x_1^{\tilde{\nu}_1} x_2^{\tilde{\nu}_2} \dots x_n^{\tilde{\nu}_n}.$$

Let $C_n^{(e)}$ denote the number of generalized compositions $\tilde{\alpha}$ of type e -Catalan such that $\tilde{\alpha} = \tilde{\nu} 0 0 \dots$ and $\ell(\tilde{\nu}) = n$. These are in bijection with the paths from $(0, 0)$ to $(n + e, n)$ that remain above the line $y = x - e$. Indeed, if we have a path of type e -Catalan, it suffices to add a horizontal line from $(\tilde{\nu}_1 + \tilde{\nu}_2 + \dots + \tilde{\nu}_n, n)$ to $(n + e, n)$. When $e = 0$, we have $C_n^{(0)} = C_n$ the n th Catalan number. This enumerates the classical Dyck paths from $(0, 0)$ to (n, n) remaining above the line $y = x$. See [16] for an extensive account on Catalan numbers. We have the following generalization to our Theorem 1.2.

Corollary 4.3. $\dim(R_n^{(e)}) \leq C_n^{(e)}$.

Proof. We use the same argument as in Lemma 4.2. For this we use the fact that for any monomial $x_1^{\tilde{\nu}_1} x_2^{\tilde{\nu}_2} \dots x_n^{\tilde{\nu}_n}$ in $\mathbb{Q}[x_1, x_2, \dots, x_n]$, if $\tilde{\nu}$ reaches level e , then $G_{\tilde{\nu} 0 \dots}(x_1, x_2, \dots, x_n) \in \mathcal{J}_n^{(e)}$. Hence a basis of $R_n^{(e)}$ is contained in the monomials corresponding to paths of type e -Catalan and our result follows. \square

Again we expect the equality to hold and we will address this question in [2].

5. IT IS A HILBERT BASIS

In the previous Section, the generating set $\mathcal{Q}^{(e)} = \{G_{\tilde{\alpha}}\}_{\tilde{\alpha} \in \mathcal{G}^{(e)}}$ of $\mathcal{J}^{(e)}$ was used to reduce every monomial to e -Catalan type generalized compositions. It is in fact a Hilbert basis for the given ideal. We use here ideas of Gröbner basis theory. This is crucial to complete the proof of Theorem 1.1. Let us recall a few basic facts about Gröbner bases; see [8] for more details.

To show that a set S is a Gröbner basis it is enough to show that all polynomial syzygies of that set are reducible in S . The polynomial syzygy of P and Q is defined by $S(P, Q) = LC(Q)M_1P - LC(P)M_2Q$ where $\text{lcm}(LM(P), LM(Q)) = M_1 \cdot LM(P) = M_2 \cdot LM(Q)$. This shows that the given set contains all the generators of the leading monomials of the ideal. We use the classic Buchberger's lemma [8]:

Lemma 5.1. *Given $P, Q \in S$. If there is an $R \in S$ such that $LM(R)$ divides $\text{lcm}(LM(P), LM(Q))$, and if both $S(R, Q)$ and $S(R, P)$ are reducible in S , then $S(P, Q)$ is reducible in S . \square*

This result is easily adapted to our context. We first remark that our sets $\mathcal{Q}^{(e)}$ satisfies the following useful property:

Lemma 5.2. *For all $\tilde{\alpha} \leq \tilde{\rho}$ (componentwise),*

$$G_{\tilde{\alpha}} \in \mathcal{Q}^{(e)} \quad \implies \quad G_{\tilde{\rho}} \in \mathcal{Q}^{(e)}$$

Proof. If $\tilde{\alpha} = \tilde{\pi}\tilde{\nu}$ satisfies $d(\tilde{\pi}) - \ell(\tilde{\pi}) \geq e$, then let $r = \ell(\tilde{\pi})$ and consider $\tilde{\rho} = \tilde{\gamma}\tilde{\mu}$ where $\ell(\tilde{\gamma}) = r$. Since $d(\tilde{\pi}) \leq d(\tilde{\gamma})$, we have $d(\tilde{\gamma}) - \ell(\tilde{\gamma}) \geq d(\tilde{\pi}) - \ell(\tilde{\pi}) \geq e$. By Lemma 3.1, $G_{\tilde{\rho}} \in \mathcal{Q}^{(e)}$. \square

We can now adapt the proof (see [8]) of Lemma 5.1 to our situation. For any pair $\tilde{\alpha}, \tilde{\pi} \in \mathcal{G}^{(e)}$, we define $S(G_{\tilde{\alpha}}, G_{\tilde{\pi}})$ as before with our definition of LM and LC . We show in this section that any such $S(G_{\tilde{\alpha}}, G_{\tilde{\pi}})$ is reducible in $\mathcal{Q}^{(e)}$. Let $\tilde{\rho} \in \mathcal{G}^{(e)}$ be the unique element such that $X^{\tilde{\rho}} = \text{lcm}(X^{\tilde{\alpha}}, X^{\tilde{\pi}}) = M_1X^{\tilde{\alpha}} = M_2X^{\tilde{\pi}}$. We have

$$(5.1) \quad \begin{aligned} S(G_{\tilde{\alpha}}, G_{\tilde{\pi}}) &= M_1G_{\tilde{\alpha}} - M_2G_{\tilde{\pi}} = M_1G_{\tilde{\alpha}} - G_{\tilde{\rho}} + G_{\tilde{\rho}} - M_2G_{\tilde{\pi}} \\ &= S(G_{\tilde{\alpha}}, G_{\tilde{\rho}}) + S(G_{\tilde{\rho}}, G_{\tilde{\pi}}). \end{aligned}$$

If both $S(G_{\tilde{\alpha}}, G_{\tilde{\rho}})$ and $S(G_{\tilde{\rho}}, G_{\tilde{\pi}})$ are reducible in $\mathcal{Q}^{(e)}$, then so is $S(G_{\tilde{\alpha}}, G_{\tilde{\pi}})$. It is sufficient to show that all $S(G_{\tilde{\alpha}}, G_{\tilde{\rho}})$ are reducible in $\mathcal{Q}^{(e)}$ for $\tilde{\alpha} \leq \tilde{\rho}$ componentwise.

We can reduce further our problem as follows. Assume that $\tilde{\alpha}$ and $\tilde{\rho}$ in $\mathcal{G}^{(e)}$ are generalized compositions of d_1 and d_2 respectively. If $\tilde{\alpha} \leq \tilde{\rho}$, then $d_1 \leq d_2$. If $d_2 - d_1 > 1$, we can select a generalized composition $\tilde{\alpha} \leq \tilde{\pi} \leq \tilde{\rho}$ and use (5.1) again. We can thus assume that $d_2 - d_1 = 1$. That is the two generalized compositions differ on one part by one unit.

Lemma 5.3. *The set $\mathcal{Q}^{(e)}$ is a Gröbner basis of $\mathcal{J}^{(e)}$.*

Proof. From the discussion above it is sufficient to show that all the expressions of $\mathcal{J}^{(e)}$ of the form $S(G_{\tilde{\gamma}a\tilde{\beta}}, G_{\tilde{\gamma}(a-1)\tilde{\beta}}) = G_{\tilde{\gamma}a\tilde{\beta}} - x_k G_{\tilde{\gamma}(a-1)\tilde{\beta}}$ where $k = \ell(\tilde{\gamma}) + 1$, are reducible in $\mathcal{Q}^{(e)}$. Let us denote by $m_{\tilde{\gamma}a\tilde{\beta}}(x_1, x_2, \dots)$ the leading monomial of $S(G_{\tilde{\gamma}a\tilde{\beta}}, G_{\tilde{\gamma}(a-1)\tilde{\beta}})$.

Let $\tilde{\beta} = \tilde{\nu}00\dots$. We set up an induction on $\ell(\tilde{\nu})$. Assume first that $\tilde{\nu} = \nu$ is a (possibly empty) standard composition. The second part of Lemma 3.1 and the recursive definition (3.2) give $S(G_{\tilde{\gamma}a\nu0\dots}, G_{\tilde{\gamma}(a-1)\nu0\dots}) = G_{\tilde{\gamma}0a\nu0\dots} \in \mathcal{Q}^{(e)}$. If $\tilde{\nu}$ is not standard, then let $\tilde{\beta} = \tilde{\pi}0b\mu00\dots$ for $b > 0$ and μ a (possibly empty) standard composition. Let $\ell = \ell(\tilde{\gamma}a\tilde{\pi}) + 1$. Using (3.2), we have

$$\begin{aligned} S(G_{\tilde{\gamma}a\tilde{\pi}0b\mu0\dots}, G_{\tilde{\gamma}(a-1)\tilde{\pi}0b\mu0\dots}) &= G_{\tilde{\gamma}a\tilde{\pi}0b\mu0\dots} - x_k G_{\tilde{\gamma}(a-1)\tilde{\pi}0b\mu0\dots} \\ &= G_{\tilde{\gamma}a\tilde{\pi}b\mu0\dots} - x_\ell G_{\tilde{\gamma}a\tilde{\pi}(b-1)\mu0\dots} \\ &\quad - x_k (G_{\tilde{\gamma}(a-1)\tilde{\pi}b\mu0\dots} - x_\ell G_{\tilde{\gamma}(a-1)\tilde{\pi}(b-1)\mu0\dots}) \\ &= S(G_{\tilde{\gamma}a\tilde{\pi}b\mu0\dots}, G_{\tilde{\gamma}(a-1)\tilde{\pi}b\mu0\dots}) \\ &\quad - x_\ell S(G_{\tilde{\gamma}a\tilde{\pi}(b-1)\mu0\dots}, G_{\tilde{\gamma}(a-1)\tilde{\pi}(b-1)\mu0\dots}). \end{aligned}$$

We observe that $m_{\tilde{\gamma} a \tilde{\pi} 0 b \mu 0 \dots}(x_1, x_2, \dots) = x_\ell m_{\tilde{\gamma} a \tilde{\pi}(b-1)\mu 0 \dots}(x_1, x_2, \dots)$ since $m_{\tilde{\gamma} a \tilde{\pi} b \mu 0 \dots}(x_1, x_2, \dots)$ is distinct and lexicographically smaller than $x_\ell m_{\tilde{\gamma} a \tilde{\pi}(b-1)\mu 0 \dots}(x_1, x_2, \dots)$. Thus $S(G_{\tilde{\gamma} a \tilde{\pi} 0 b \mu 0 \dots}, G_{\tilde{\gamma}(a-1)\tilde{\pi} 0 b \mu 0 \dots})$ is resolved by two expressions such that $\ell(\tilde{\pi} b \mu) = \ell(\tilde{\pi}(b-1)\mu) < \ell(\tilde{\nu})$ and by the induction hypothesis $S(G_{\tilde{\gamma} a \tilde{\pi} 0 b \mu 0 \dots}, G_{\tilde{\gamma}(a-1)\tilde{\pi} 0 b \mu 0 \dots})$ is reducible in $\mathcal{Q}^{(e)}$. \square

Corollary 5.4. *The set $\mathcal{Q}^{(e)}$ is a Hilbert basis of $\overline{\mathcal{J}^{(e)}}$.*

Proof. Given an element $P \in \overline{\mathcal{J}^{(e)}}$ let $X^{\tilde{\beta}} = LM(P)$. Since $\mathcal{Q}^{(e)}$ is a Gröbner basis it contains an element $G_{\tilde{\alpha}}$ such that $\tilde{\alpha} \leq \tilde{\beta}$ componentwise. Lemma 5.2 gives us that $G_{\tilde{\beta}} \in \mathcal{Q}^{(e)}$. The element $P - LC(P) \cdot G_{\tilde{\alpha}} \in \overline{\mathcal{J}^{(e)}}$ is such that $LM(P - LC(P) \cdot G_{\tilde{\alpha}}) <_{lex} X^{\tilde{\beta}}$. If we repeat this process (possibly countably many times) we can express P as a series in the elements of $\mathcal{Q}^{(e)}$. \square

We are now in position to conclude our investigation and prove the general version of our Theorem 1.1.

Corollary 5.5. *the monomial Hilbert basis of $R^{(e)}$ is given by the monomials of $\mathbb{Q}[[x_1, x_2, x_3, \dots]]$ corresponding to the paths of type e -Catalan.*

Proof. As noted in Remark 3.5, the set $\{G_{\tilde{\alpha}}\}$ forms a Hilbert basis of $\mathbb{Q}[[x_1, x_2, x_3, \dots]]$, and Corollary 5.4 gives that the set $\mathcal{Q}^{(e)}$ is a Hilbert basis of $\overline{\mathcal{J}^{(e)}}$. Thus a Hilbert basis of the quotient $R^{(e)} = \mathbb{Q}[[x_1, x_2, x_3, \dots]]/\overline{\mathcal{J}^{(e)}}$ is given by the set $\{G_{\tilde{\alpha}}\} \setminus \mathcal{Q}^{(e)} = \{G_{\tilde{\alpha}} \mid \tilde{\alpha} \text{ corresponds to a path of type } e\text{-Catalan}\}$. The result follows by triangularity. \square

Remark 5.6. To show the equality in Theorem 1.2, it appears that the set $\{G_{\tilde{\alpha}} \mid \ell(\tilde{\alpha}) = n, \text{ and } \tilde{\alpha} \text{ reaches level } e\}$ forms a linear basis of $J_n^{(e)}$. Unfortunately the argument of Section 5 is not sufficient to show this with finitely many variables and it requires more work. This is the object of our collaboration in [2].

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REFERENCES

- [1] E. ARTIN, *Galois Theory*, Notre Dame Mathematical Lecture **2** (1944), Notre Dame, IN.
- [2] J.-C. AVAL, F. BERGERON AND N. BERGERON, *Ideals of Quasi-symmetric polynomials and related varieties, part I*, in preparation.
- [3] J.-C. AVAL, F. BERGERON, N. BERGERON AND A. GARSIA, *Ideals of Quasi-symmetric polynomials and related varieties, part II*, in preparation.
- [4] F. BERGERON, N. BERGERON, A. GARSIA, M. HAIMAN AND G. TESLER, *Lattice Diagram Polynomials and Extended Pieri Rules*, Adv. Math., **142** (1999), 244-334.
- [5] F. BERGERON, A. GARSIA AND G. TESLER, *Multiple Left Regular Representations Generated by Alternants*, J. of Comb. Th., Series A, **91**, **1-2** (2000), 49-83.
- [6] N. BERGERON, S. MYKYTIUK, F. SOTTILE, AND S. VAN WILLIGENBURG, *Pieri Operations on Posets*, J. of Comb. Theory, Series A, **91** (2000), 84-110 .
- [7] C. DE CONCINI AND C. PROCESI, *Symmetric functions, conjugacy classes and the flag variety*, Invent. Math. **64** (1981), 203-230.
- [8] D. COX, J. LITTLE AND D. O'SHEA, *Ideals, Varieties, and Algorithms*, Springer-Verlag, New-York, 1992.

- [9] A. M. GARSIA AND M. HAIMAN, *A graded representation model for Macdonald's polynomials*, Proc. Nat. Acad. Sci. U.S.A. **90** (1993), no. 8, 3607–3610.
- [10] I. GELFAND, D. KROB, A. LASCoux, B. LECLERC, V. RETAKH, AND J.-Y. THIBON, *Noncommutative symmetric functions*, Adv. in Math., **112** (1995), 218–348.
- [11] I. GESSEL, *Multipartite P-partitions and products of skew Schur functions*, in Combinatorics and Algebra (Boulder, Colo., 1983), C. Greene, ed., vol. 34 of Contemp. Math., AMS, 1984, pp. 289–317.
- [12] M. HAIMAN, *Hilbert schemes, polygraphs, and the Macdonald positivity conjecture*, J. Amer. Math. Soc. **14** (2001), 941–1006.
- [13] F. HIVERT, *Hecke algebras, difference operators, and quasi-symmetric functions*, Adv. Math. **155** (2000), 181–238.
- [14] I. MACDONALD, *Symmetric Functions and Hall Polynomials*, Oxford Univ. Press, 1995, second edition.
- [15] C. MALVENUTO AND C. REUTENAUER, *Duality between quasi-symmetric functions and the Solomon descent algebra*, Journal of Algebra, **177** (1995), 967–982.
- [16] R. STANLEY, *Enumerative Combinatorics, Vol. 1*, Wadsworth and Brooks/Cole, 1986.
- [17] R. STANLEY, *Enumerative Combinatorics Vol. 2*, no. 62 in Cambridge Studies in Advanced Mathematics, Cambridge University Press, 1999. Appendix 1 by Sergey Fomin.
- [18] R. STEINBERG, *Differential equations invariant under finite reflection groups*, Trans. Amer. Math. Soc. **112** (1964), 392–400.

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