

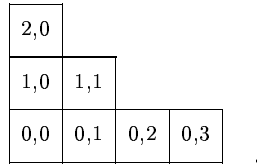
VANISHING IDEALS OF LATTICE DIAGRAM DETERMINANTS

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ABSTRACT. A lattice diagram is a finite set $L = \{(p_1, q_1), \dots, (p_n, q_n)\}$ of lattice cells in the positive quadrant. The corresponding lattice diagram determinant is $\Delta_L(X_n; Y_n) = \det \|x_i^{p_j} y_i^{q_j}\|$. The space M_L is the space spanned by all partial derivatives of $\Delta_L(X_n; Y_n)$. We denote by M_L^0 the Y -free component of M_L . For μ a partition of $n + 1$, we denote by μ/ij the diagram obtained by removing the cell (i, j) from the Ferrers diagram of μ . Using homogeneous partially symmetric polynomials, we give here a dual description of the vanishing ideal of the space M_μ^0 and we give the first known description of the vanishing ideal of $M_{\mu/ij}^0$.

1. INTRODUCTION

A *lattice diagram* is a finite set $L = \{(p_1, q_1), \dots, (p_n, q_n)\}$ of lattice cells in the positive quadrant. Following the definitions and conventions of [5, 7], the coordinates $p_i \geq 0$ and $q_i \geq 0$ of a cell (p_i, q_i) indicate the row and column position, respectively, of the cell in the positive quadrant. For $\mu_1 \geq \mu_2 \geq \dots \geq \mu_k > 0$, we say that $\mu = (\mu_1, \mu_2, \dots, \mu_k)$ is a *partition* of n if $n = \mu_1 + \dots + \mu_k$. We associate to a partition μ the following lattice (Ferrers) diagram $\{(i, j) : 0 \leq i \leq k - 1, 0 \leq j \leq \mu_{i+1} - 1\}$ and we use the symbol μ for both the partition and its associated Ferrers diagram. For example, given the partition $(4, 2, 1)$, its Ferrers diagram is:



This consists of the lattice cells $\{(0, 0), (1, 0), (2, 0), (0, 1), (1, 1), (0, 2), (0, 3)\}$. We list the cells in *lexicographic* order looking at the second-coordinate first. That is:

$$(1.1) \quad (p_1, q_1) < (p_2, q_2) \iff q_1 < q_2 \quad \text{or} \quad [q_1 = q_2 \text{ and } p_1 < p_2].$$

Given a lattice diagram $L = \{(p_1, q_1), (p_2, q_2), \dots, (p_n, q_n)\}$ we define the *lattice diagram determinant*

$$\Delta_L(X_n; Y_n) = \det \left\| \frac{x_i^{p_j} y_i^{q_j}}{p_j! q_j!} \right\|_{i,j=1}^n,$$

where $X_n = x_1, x_2, \dots, x_n$ and $Y_n = y_1, y_2, \dots, y_n$. The determinant $\Delta_L(X_n; Y_n)$ is bihomogeneous of degree $|p| = p_1 + \dots + p_n$ in X_n and degree $|q| = q_1 + \dots + q_n$ in Y_n . To ensure that this definition associates a unique determinant to L we require that

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the list of lattice cells be ordered with the lexicographic order 1.1. The factorials will ensure that the lattice diagram determinants behave nicely under partial derivatives.

For a polynomial $P(X_n; Y_n)$, the vector space spanned by all the partial derivatives of P of all orders is denoted $\mathcal{L}_\partial[P]$. A permutation $\sigma \in \mathcal{S}_n$ acts diagonally on a polynomial $P(X_n; Y_n)$ as follows: $\sigma P(X_n; Y_n) = P(x_{\sigma_1}, x_{\sigma_2}, \dots, x_{\sigma_n}; y_{\sigma_1}, y_{\sigma_2}, \dots, y_{\sigma_n})$. Under this action, $\Delta_L(X_n; Y_n)$ is clearly an alternant. Moreover, partial derivatives commute with the action, hence it follows that for any lattice diagram L with n cells, the vector space $M_L = \mathcal{L}_\partial[\Delta_L(X_n; Y_n)]$ is an \mathcal{S}_n -module. Since $\Delta_L(X_n; Y_n)$ is bihomogeneous, this module affords a natural bigrading. Denoting by $\mathcal{H}_{r,s}[M_L]$ the subspace consisting of the bihomogeneous elements of degree r in X_n and degree s in Y_n , we have the direct sum decomposition

$$M_L = \bigoplus_{r=0}^{|p|} \bigoplus_{s=0}^{|q|} \mathcal{H}_{r,s}[M_L].$$

In general, not much is known on the \mathcal{S}_n -modules M_L . In the case where L is a Ferrers diagram μ with n cells, the $n!$ conjecture of Garsia and Haiman [10] stated that M_μ has dimension $n!$, that it is \mathcal{S}_n -isomorphic to the left regular representations and its graded character is a renormalization of the Macdonald polynomial [13] indexed by μ . Haiman in [12] has recently completed a proof of this using an algebraic geometry approach. It develops that a very natural and combinatorial recursive approach to the result above involves diagrams obtained by removing a single cell from a partition diagram. In [5], we have investigated this case and formulated various new conjectures.

To pursue the investigation of the spaces M_L , we are interested in an explicit description of the vanishing ideal I_L of differential operators on the spaces M_L , which is defined in the following way:

$$I_L = \{P(X_n; Y_n) \in \mathbb{Q}[X_n; Y_n] : P(\partial X_n; \partial Y_n) \Delta_L(X_n; Y_n) = 0\},$$

where for a polynomial $P(X_n; Y_n)$ we denote by $P(\partial X_n; \partial Y_n)$ the differential operator obtained from P , substituting every variable x_i by the operator $\frac{\partial}{\partial x_i}$ and every variable y_j by the operator $\frac{\partial}{\partial y_j}$. A step in this program is to describe the vanishing ideal of a special subspace of M_L . For a lattice diagram L , we let

$$M_L^0 = \bigoplus_{r=0}^{|p|} \mathcal{H}_{r,0}[M_L].$$

That is the Y -free component of M_L . In this paper, we study the vanishing ideal I_L^0 of differential operator on the spaces M_L^0 . Our result gives a good set of generators for the ideal I_L^0 in the case where L is a partition μ or a partition with a hole μ/ij . The case $L = \mu$ was studied extensively in [7, 8, 11, 15]. Our description is dual to Tanisaki's [7, 15]. For $L = \mu/ij$, there was no previously known description of $I_{\mu/ij}^0$.

Our analysis is based on a careful study of the effect of partially symmetric differential operators on lattice diagram determinants. We do this in Section 3. For completeness, we also give a description for Schur symmetric differential operators.

In Section 4 we recall the known results for I_μ^0 and give the dual description using homogeneous partially symmetric polynomials. In Section 5 we describe $I_{\mu/ij}^0$.

2. SOME PRELIMINARY RESULTS

Before we start, let us collect some useful facts. We need a few definitions. For an n -cell lattice diagram L , a *tableau* of shape L is an injective map $T : L \rightarrow \{1, 2, \dots, n\}$. We can think of T as a way to list the cells of L . If $T(r, c) = m$, we say that $h_T(m) = r$ is the *height* of m in T . We say that T is *column increasing* if $T(r, c_1) < T(r, c_2)$ whenever $c_1 < c_2$ (when this has a meaning). Let \mathcal{T}_L be the set of all tableaux of shape L and let \mathcal{CT}_L be the set of all column increasing tableaux of shape L . Finally, for any tableau T of shape L , we let

$$\Gamma(T) = \left\{ \{T(r, k) \mid (r, k) \in L\} \mid 0 \leq k \leq \max_{(r,c) \in L} c \right\}$$

be the *column sets* of T .

We now expand the determinant

$$\Delta_L(X_n, Y_n) = \left(\prod_{(r,c) \in L} \frac{1}{r! c!} \right) \sum_{T \in \mathcal{T}_L} \pm m_T(Y_n) \tilde{m}_T(X_n)$$

where

$$m_T(Y_n) = \prod_{(r,c) \in L} y_{T(r,c)}^c, \quad \tilde{m}_T(X_n) = \prod_{(r,c) \in L} x_{T(r,c)}^r,$$

and the sign is the sign of the permutation that reorders the cells of L given by T back in the lexicographic order 1.1. If we now collect the terms of $\Delta_L(X_n, Y_n)$ with the same monomials in Y_n , we easily see that

$$(2.1) \quad \Delta_L(X_n, Y_n) = \left(\prod_{(r,c) \in L} \frac{1}{r! c!} \right) \sum_{T \in \mathcal{CT}_L} m_T(Y_n) \Delta_T(X_n)$$

where

$$(2.2) \quad \Delta_T(X_n) = \pm \prod_{c \in \Gamma(T)} \det \|x_m^{h_T(\ell)}\|_{m, \ell \in c}.$$

For $P(X_n) \in \mathbb{Q}[X_n]$, we have

$$(2.3) \quad P(\partial X_n) \Delta_L(X_n; Y_n) = 0 \iff P(X_n) \in I_L^0.$$

On one hand, if $P(\partial X_n) \Delta_L(X_n; Y_n) = 0$ then clearly $P(\partial X_n) Q(X_n; Y_n) = 0$ for all $Q(X_n; Y_n) \in M_L$. In particular $P(\partial X_n) Q(X_n) = 0$ for all $Q(X_n) \in M_L^0 \subseteq M_L$ and $P(X_n) \in I_L^0$. On the other hand, consider the expansion 2.1 of $\Delta_L(X_n; Y_n)$. For any column strict tableau T of shape L we have $m_T(\partial Y_n) \Delta_L(X_n; Y_n) = c \Delta_T(X_n)$ for a non-zero constant c , and $\Delta_T(X_n) \in M_L^0$. Hence if $P(X_n) \in I_L^0$, we have that $P(\partial X_n) \Delta_T(X_n) = 0$ for all T in the expansion 2.1 and thus $P(\partial X_n) \Delta_L(X_n; Y_n) = 0$.

3. SYMMETRIC OPERATORS

For the sake of simplicity, we limit our descriptions to X -operators; the Y -operators are similar. Recall that

$$\begin{aligned} \bullet p_k(X_n) &= \sum_{i=1}^n x_i^k, \\ \bullet e_k(X_n) &= \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq n} x_{i_1} x_{i_2} \cdots x_{i_k} \quad \text{and} \\ \bullet h_k(X_n) &= \sum_{1 \leq i_1 \leq i_2 \leq \dots \leq i_k \leq n} x_{i_1} x_{i_2} \cdots x_{i_k} \end{aligned}$$

are the power sum, elementary and homogeneous symmetric polynomials, respectively. In [2, 5] we have described the effect of the above symmetric differential operators on lattice diagram determinants. We now recall the results stated in [2].

In this section we will assume that a lattice diagram is a lists of cells

$$L = [(p_1, q_1), (p_2, q_2), \dots, (p_n, q_n)]$$

ordered by the order 1.1. We allow L to have repeated entries or negative coordinates, but in that case we set the determinant $\Delta_L(X_n; Y_n) = 0$. We define the function

$$(3.1) \quad \epsilon(L) = \begin{cases} 1 & \text{if } L \text{ has distinct cells in the positive quadrant,} \\ 0 & \text{otherwise.} \end{cases}$$

Let L be a lattice diagram as above with n distinct cells in the positive quadrant. For any integer $k \geq 1$ we have:

Proposition 3.1 (Proposition I.1 [5]).

$$p_k(\partial X_n) \Delta_L(X_n; Y_n) = \sum_{i=1}^n \pm \Delta_{p_k(i, L)}(X_n; Y_n)$$

where $p_k(i, L)$ is the lattice diagram obtained from L by replacing the i -th biexponent (p_i, q_i) with $(p_i - k, q_i)$, and the sign of $\Delta_{p_k(i, L)}(X_n; Y_n)$ is the sign of the permutation that reorders the resulting biexponents in the order 1.1.

Proposition 3.2 (Proposition 2 [2]).

$$e_k(\partial X_n) \Delta_L(X_n; Y_n) = \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq n} \Delta_{e_k(i_1, \dots, i_k; L)}(X_n; Y_n)$$

where $e_k(i_1, \dots, i_k; L)$ is the lattice diagram obtained from L by replacing the biexponents $(p_{i_1}, q_{i_1}), \dots, (p_{i_k}, q_{i_k})$ with $(p_{i_1} - 1, q_{i_1}), \dots, (p_{i_k} - 1, q_{i_k})$.

For a lattice diagram L with n distinct cells in the positive quadrant, we denote by \bar{L} its complement in the positive quadrant (it is an infinite subset). Again we list $\bar{L} = [(\bar{p}_1, \bar{q}_1), (\bar{p}_2, \bar{q}_2), \dots]$ using the lexicographic order 1.1.

Proposition 3.3 (Proposition 3 [2]).

$$h_k(\partial X_n) \Delta_L(X_n; Y_n) = \sum_{1 \leq i_1 < i_2 < \dots < i_k} \epsilon(e_k(i_1, \dots, i_k, \bar{L})) \Delta_{\frac{\bar{L}}{e_k(i_1, \dots, i_k, \bar{L})}}(X_n; Y_n)$$

where $e_k(i_1, \dots, i_k, \overline{L})$ is the lattice diagram obtained from \overline{L} by replacing the biexponents $(\overline{p}_{i_1}, \overline{q}_{i_1}), \dots, (\overline{p}_{i_k}, \overline{q}_{i_k})$ with $(\overline{p}_{i_1} + 1, \overline{q}_{i_1}), \dots, (\overline{p}_{i_k} + 1, \overline{q}_{i_k})$. Its complement is $e_k(i_1, \dots, i_k, \overline{L})$ and the function ϵ is defined in 3.1.

We remark that even if the sum in Proposition 3.3 is infinite, only a finite number of coefficients $\epsilon(e_k(i_1, \dots, i_k, \overline{L}))$ differs from zero. We will give at the end of this section an expression for $h_k(\partial X_n) \Delta_L(X_n; Y_n)$ that does not depend on the complement of L . For this, it is natural to ask what is the effect of a Schur differential operator on a lattice diagram determinant. For completeness, we insert here such a description that unifies Proposition 3.2 and 3.3. We only sketch a proof of our description since this is not used in the remaining sections and the technique of proofs are well known.

Following [13], recall that for a partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$ the conjugate (transpose) partition is denoted by $\lambda' = (\lambda'_1, \lambda'_2, \dots, \lambda'_\ell)$. With this in mind, the Schur polynomial indexed by λ is

$$(3.2) \quad S_\lambda(X_n) = \det \| e_{\lambda'_i + j - i}(X_n) \| = \sum_{\sigma \in \mathcal{S}_\ell} \text{sgn}(\sigma) e_{\sigma(\lambda' + \delta_\ell) - \delta_\ell},$$

with the understanding that $e_0(X_n) = 1$ and $e_k(X_n) = 0$ if $k < 0$. The Schur polynomials also have a description in terms of column-strict Young tableaux. Given λ a partition of n , a tableau of shape λ is a map $T: \lambda \rightarrow \{1, 2, \dots, n\}$. We say that T is a column-strict Young tableau if it is weakly increasing along the rows and strictly increasing along columns of λ . That is $T(i, j) \leq T(i, j+1)$ and $T(i, j) < T(i+1, j)$. We denote by \mathcal{T}_λ the set of all column-strict Young tableaux of shape λ . For any tableau T , we define $X_n^T = \prod_{i=1}^n x_i^{|T^{-1}(i)|}$. As seen in [13], we have

$$S_\lambda(X_n) = \sum_{T \in \mathcal{T}_\lambda} X_n^T.$$

Let L be a lattice diagram with n cells ordered by the order 1.1. For any partition λ of an integer $k \geq 1$ we have

Proposition 3.4.

$$S_\lambda(\partial X_n) \Delta_L(X_n; Y_n) = \sum_{T \in \mathcal{T}_\lambda} \epsilon'(T, L) \Delta_{\partial T(L)}(X_n; Y_n)$$

where $\partial T(L)$ is the lattice diagram obtained from L by replacing the biexponents (p_i, q_i) with $(p_i - |T^{-1}(i)|, q_i)$ for $1 \leq i \leq n$. The coefficient

$$\epsilon'(T, L) = \epsilon(\partial T(L)) \cdots \epsilon(\partial T_{\ell-1} \partial T_\ell(L)) \epsilon(\partial T_\ell(L)),$$

where T_1, T_2, \dots, T_ℓ are the ℓ columns of T and ϵ is the function defined in 3.1.

There are many standard ways to prove this statement. For example one can iteratively use the Proposition 3.2 with the expansion 3.2. Then, Proposition 3.4 follows by a suitable canceling involution. See [14] for a similar involution. A complete proof of this Proposition will appear elsewhere [4].

Remark 3.5. Given a lattice diagram L and a column strict tableau $T \in \mathcal{T}_\lambda$, we have that $\epsilon'(T, L) = 1$ exactly when we can move the cells of L by one, reading T column by column, from right to left, without having any cells colliding.

Corollary 3.6. *For $h_k(X_n) = s_{(k)}(X_n)$ we have*

$$h_k(\partial X_n) \Delta_L(X_n; Y_n) = \sum_{1 \leq i_1 \leq i_2 \leq \dots \leq i_k \leq n} \epsilon'((i_1, \dots, i_k), L) \Delta_{\partial_{i_1} \dots \partial_{i_k}(L)}(X_n; Y_n)$$

This is equivalent to the description in Proposition 3.3. The only way that $\epsilon'((i_1, \dots, i_k), L) \neq 0$ is if the cells i_1, \dots, i_k that move down are moved into holes. This can be described as distinct holes moving up.

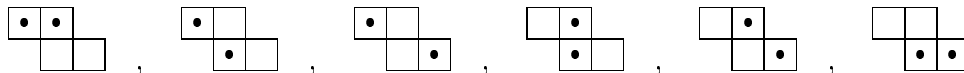
For the following sections we now describe a necessary condition that tests if a partially symmetric operator belongs to the vanishing ideal of a lattice diagram determinant. For $k \leq n$, fix

$$S = \{x_{i_1}, x_{i_2}, \dots, x_{i_k}\} \subseteq X_n,$$

and set $S^Y = \{y_{i_1}, y_{i_2}, \dots, y_{i_k}\}$. Given L a lattice diagram with n cells, we can expand the determinant $\Delta_L(X_n; Y_n)$ in term of the rows i_1, i_2, \dots, i_k and we get

$$(3.3) \quad \Delta_L(X_n; Y_n) = \sum_{\substack{D \subseteq L, \\ |D|=k}} \pm \Delta_D(S; S^Y) \Delta_{L-D}(X_n - S, Y_n - S^Y).$$

Clearly, if a symmetric operator in the variables S annihilates all $\Delta_D(S; S^Y)$ for $D \subseteq L$, then it annihilates $\Delta_L(X_n; Y_n)$. For example, let $L = \{(1, 0), (0, 1), (1, 1), (0, 2)\}$ and choose $S \subseteq X_n$ such that $|S| = k = 2$. We have that $h_3(S) \in I_L^0$, since for all subsets $D \subseteq L$ such that $|D| = 2$ the Proposition 3.3 gives us that $h_3(\partial S) \Delta_D(S; S^Y) = 0$. We can visualize this as follows. We represent all subsets D putting two \bullet in the all possible ways inside L .



The maximal number of distinct cells in \overline{D} that can go up without overlapping another in all pictures is two. For example, in the first picture, there are two cells below the two \bullet , if we try to move up three cells of \overline{D} , necessarily at least two will overlap and hence $h_3(\partial S) \Delta_D(S; S^Y) = 0$. Using this technique we can easily check that $h_r(S) \in I_L^0$ if $r \geq 2$ for $|S| = 1$, $r \geq 3$ for $|S| = 2$, $r \geq 3$ for $|S| = 3$ and $r \geq 2$ for $|S| = 4$.

4. THE IDEAL I_μ^0

In this section we recall the results [7, 15] for the ideal I_μ^0 and give a dual description of the ideal in terms of homogeneous partially symmetric polynomials. This gives us a better understanding for the case I_{μ/i_j}^0 in Section 5.

Let μ be a fixed partition of n . The homogeneous partially symmetric polynomials are the polynomials $h_r(S)$ for $S = \{x_{i_1}, x_{i_2}, \dots, x_{i_k}\} \subseteq X_n$. Let $\mu' = (\mu'_1, \mu'_2, \dots, \mu'_m)$ denote the conjugate partition of μ . For $1 \leq k \leq n$, we define

$$\delta_k(\mu) = \mu'_1 + \mu'_2 + \dots + \mu'_k$$

with the convention that $\mu'_j = 0$ if $j > m$.

Proposition 4.1. *For μ a partition of n*

$$I_\mu^0 = \langle h_r(S) : S \subseteq X_n, |S| = k, r > \delta_k(\mu) - k \rangle.$$

Tanisaki [15], with a simplified proof in [7], shows that

$$\begin{aligned} I_\mu^0 &= \langle e_r(\overline{S}) : \overline{S} \subseteq X_n, |\overline{S}| = \overline{k}, r > \overline{k} - (n - \delta_{n-\overline{k}}(\mu)) \rangle \\ &= \langle e_r(\overline{S}) : \overline{S} \subseteq X_n, |\overline{S}| = n - k, r > \delta_k(\mu) - k \rangle. \end{aligned}$$

In light of the following lemma, the Proposition 4.1 is simply the dual description of Tanisaki's description of I_μ^0 . More precisely, using the ideal

$$I_{1^n}^0 = \langle e_r(X_n) : r > 0 \rangle = \langle h_r(X_n) : r > 0 \rangle \subseteq I_\mu^0,$$

we have:

Lemma 4.2. *For $S \subseteq X_n$ let $\overline{S} = X_n - S$, then*

$$h_r(S) \equiv (-1)^r e_r(\overline{S}) \pmod{I_{1^n}^0}.$$

Proof. We know that $e_r(\overline{S})$ and $h_r(S)$ are the coefficient of t^r in

$$E_{\overline{S}}(t) = \prod_{i \in \overline{S}} (1 + tx_i) \quad \text{and} \quad H_S(t) = \prod_{i \in S} \frac{1}{1 - tx_i}$$

respectively. Since $S \cup \overline{S} = X_n$ is a disjoint union, we have that $E_{\overline{S}}(t)/H_S(-t) = E_{X_n}(t) \equiv 1 \pmod{I_{1^n}^0}$. Hence $E_{\overline{S}}(t) \equiv H_S(-t) \pmod{I_{1^n}^0}$ and the result follows. \square

Remark 4.3. The reader should note that the argument of [7, 15] could be further simplified using directly the generators of Proposition 4.1. As described at the end of Section 3, one can easily show that $h_r(\partial S)\Delta_\mu(X_n; Y_n) = 0$ whenever $|S| = k$ and $r > \delta_k(\mu) - k$. This is a simple use of the pigeon hole principle. Then the reduction algorithm in the proof of Proposition 4.2 [7] is better suited to the homogeneous functions $h_k(S)$.

5. THE IDEAL $I_{\mu/ij}^0$

We describe the vanishing ideal $I_{\mu/ij}^0$ of the space $M_{\mu/ij}^0$ for μ a partition of $n + 1$ and (i, j) a cell of μ . In the previous section we have seen that I_μ^0 has two dual descriptions in terms of elementary partially symmetric function and in terms of homogeneous partially symmetric functions. The key idea was that they are both equivalent modulo the ideal $I_{1^n}^0$ of fully symmetric functions that is contained in every I_μ^0 . To study $I_{\mu/ij}^0$ we do not have $I_{1^n}^0 \subseteq I_{\mu/ij}^0$. For example, if $(i, j) \in \mu$ is not at the top of a column of μ then $h_1(\partial X_n)\Delta_{\mu/ij}(X_n; Y_n) = \Delta_{\mu/i+1,j}(X_n; Y_n) \neq 0$. On the other hand, as we will see in Lemma 5.2, $h_r(X_n) \in I_{\mu/ij}^0$ for all other $r > 1$. To describe $I_{\mu/ij}^0$ we need to use both families of generators. Let us introduce some notation.

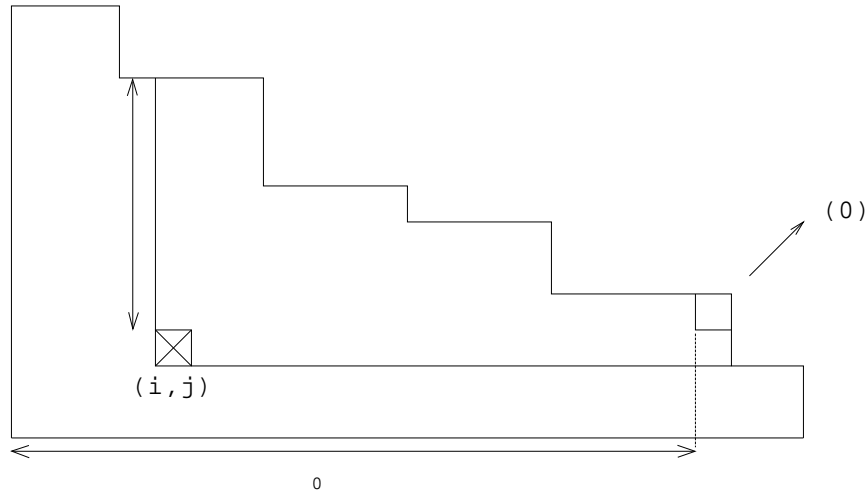


Figure 1

Let ℓ be the number of cells above the cell (i, j) in μ . In the area north east of the cell (i, j) , let (α_0, β_0) be the coordinate of the rightmost corner cell of μ . We let $\nu(0) = \mu/\alpha_0\beta_0$ be the partition of n obtained from μ by removing the cell (α_0, β_0) .

If (i, j) is at the top of a column of μ , then $\alpha_0 = i$ and

$$h_1^{\beta_0-j}(\partial Y_n)\Delta_{\mu/ij}(X_n; Y_n) = \Delta_{\mu/\alpha_0\beta_0}(X_n; Y_n) = \Delta_{\nu(0)}(X_n; Y_n).$$

If $P(\partial X_n)\Delta_{\mu/ij}(X_n; Y_n) = 0$, then clearly $P(\partial X_n)\Delta_{\nu(0)}(X_n; Y_n) = 0$ and $I_{\mu/ij}^0 \subseteq I_{\nu(0)}^0$. To see the other inclusion, we first compare the expansions 2.1 for $\Delta_{\mu/ij}(X_n; Y_n)$ and $\Delta_{\nu(0)}(X_n; Y_n)$. We have

$$\begin{aligned} \Delta_{\mu/ij}(X_n; Y_n) &= \sum_{T \in \mathcal{CT}_{\mu/ij}} m_T(Y_n)\Delta_T(X_n) \\ \Delta_{\nu(0)}(X_n; Y_n) &= \sum_{T' \in \mathcal{CT}_{\nu(0)}} m_{T'}(Y_n)\Delta_{T'}(X_n) \end{aligned}$$

There is an obvious bijection $\mathcal{CT}_{\mu/ij} \rightarrow \mathcal{CT}_{\nu(0)}$ that preserves the column set $\Gamma(T) = \Gamma(T')$. Under this pairing $T \leftrightarrow T'$ we have that $\Delta_T(X_n) = \pm\Delta_{T'}(X_n)$. If $P(\partial X_n)\Delta_{\nu(0)}(X_n; Y_n) = 0$ we must have that $0 = P(\partial X_n)\Delta_{T'}(X_n) = P(\partial X_n)\Delta_T(X_n)$ for all $T \leftrightarrow T'$. It follows that $P(\partial X_n)\Delta_{\mu/ij}(X_n; Y_n) = 0$ and this shows the inclusion $I_{\nu(0)}^0 \subseteq I_{\mu/ij}^0$. Hence if (i, j) is at the top of a column of μ , we have $I_{\mu/ij}^0 = I_{\nu(0)}^0$, a case covered in the previous section. For the remaining of this section we can thus assume that (i, j) is not at the top of a column of μ .

For μ a partition of $n + 1$ let

$$J_\mu^0 = \langle h_r(S) : S \subseteq X_n, |S| = k, r > \delta_k(\mu) - k, \text{ for } 1 \leq k \leq n \rangle.$$

In particular $h_1(X_n) \notin J_\mu$, but we clearly have

$$J_\mu^0 \subseteq \langle h_r(S) : S \subseteq X_n, |S| = k, r > \delta_k(\mu) - k, \text{ for } 1 \leq k \leq n + 1 \rangle.$$

Thus from Proposition 4.1 it follows that $J_\mu^0 \subseteq I_\mu^0$. We have seen above that the use of homogeneous partially symmetric functions may not be enough to describe $I_{\mu/ij}^0$.

In our main theorem below, we also need elementary partially symmetric functions. Let

$$\begin{aligned} \widehat{J}_{\mu/ij}^0 &= \langle h_1(X_n) e_r(\overline{S}) : \overline{S} \subseteq X_n, |\overline{S}| = n - k, j < k \leq \beta_0, r = \delta_k(\mu) - k \rangle \\ &+ \langle e_r(\overline{S}) : \overline{S} \subseteq X_n, |\overline{S}| = n - k, \beta_0 < k < \mu_1, r = \delta_k(\mu) - k \rangle. \end{aligned}$$

Theorem 5.1. *Using the notation above for $\mu = (\mu_1, \mu_2, \dots)$ a partition of $n + 1$ and (i, j) a cell of μ ,*

$$(5.1) \quad I_{\mu/ij}^0 = J_{\mu}^0 + \langle h_1^{\ell+1}(X_n) \rangle + \widehat{J}_{\mu/ij}^0.$$

The remaining of this section is dedicated to a proof of this theorem. Let $\widetilde{I}_{\mu/ij}^0$ denote the right hand side of Equation 5.1. We first show the inclusion $\widetilde{I}_{\mu/ij}^0 \subseteq I_{\mu/ij}^0$, showing that each of the components of $\widetilde{I}_{\mu/ij}^0$ belongs to $I_{\mu/ij}^0$.

Lemma 5.2. *For any $(i, j) \in \mu$, we have $J_{\mu}^0 \subseteq I_{\mu/ij}^0$.*

Proof. It is clear from the definition that $J_{\mu}^0 \subseteq I_{\mu}^0$. Let us use the Equation 3.3 with μ and x_{n+1} :

$$\Delta_{\mu}(X_n, x_{n+1}; Y_n, y_{n+1}) = \sum_{(i,j) \in \mu} \pm x_{n+1}^i y_{n+1}^j \Delta_{\mu/ij}(X_n; Y_n).$$

For any $P(X_n) \in J_{\mu}^0 \subseteq I_{\mu}^0$ we have

$$0 = P(\partial X_n) \Delta_{\mu}(X_n, x_{n+1}; Y_n, y_{n+1}) = \sum_{(i,j) \in \mu} \pm x_{n+1}^i y_{n+1}^j P(\partial X_n) \Delta_{\mu/ij}(X_n; Y_n)$$

which shows that $P(\partial X_n) \Delta_{\mu/ij}(X_n; Y_n) = 0$ for all $(i, j) \in \mu$. \square

To show that $h_1^{\ell+1}(X_n) \in I_{\mu/ij}^0$ we use the Proposition 3.3. In $\overline{\mu/ij}$ the only cell that can move up, without overlapping another, is (i, j) . It can move up by ℓ , but not more. Moving the cell by $\ell + 1$ would be outside of μ hence overlapping another cell of $\overline{\mu/ij}$. Thus

$$h_1^{\ell+1}(\partial X_n) \Delta_{\mu/ij}(X_n; Y_n) = 0.$$

Now, for $\beta_0 < k < \mu_1$ we consider $e_r(\overline{S})$ where $\overline{S} \subseteq X_n$ of cardinality $|\overline{S}| = n - k$ and $r = \delta_k(\mu) - k$. Again we shall show that $e_r(\overline{S}) \in I_{\mu/ij}^0$. We remark that if the condition $\beta_0 < k < \mu_1$ is non-empty, then (i, j) is not on the first row of μ . We expand $\Delta_{\mu/ij}(X_n; Y_n)$ as in 3.3

$$\Delta_{\mu/ij}(X_n; Y_n) = \sum_{\substack{D \subset \mu/ij, \\ |D| = n-k}} \pm \Delta_D(\overline{S}; \overline{S}^Y) \Delta_{\mu/ij-D}(X_n - \overline{S}, Y_n - \overline{S}^Y).$$

By inspection of Figure 2 below, we note that there are $n - k$ cells in total in the two shaded areas of μ/ij . In the darker grey area there are $\delta_k(\mu) - k - 1$ cells of μ/ij . According to the Proposition 3.2, to have $e_r(\partial \overline{S}) \Delta_D(\overline{S}; \overline{S}^Y) \neq 0$, we must be able to move $r = \delta_k(\mu) - k$ distinct cells of D down. Among all the diagrams $D \subset \mu/ij$ such that $|D| = n - k$, the ones that maximize the number of distinct cells that can go down are obtained as follows. We must first choose all the cells of μ/ij that are in rows

$2, 3, \dots, \mu'_1$, and there are $n - \mu_1$ such cells. We must then choose $\mu_1 - k$ cells in the first row. Each cell in the first row prevents the cells above it to move down. We must minimize these obstructions and choose cells $(0, j_1), (0, j_2), \dots, (0, j_{\mu_1 - k})$ in columns j_s such that the μ'_{j_s} are the smallest, and possibly the column j (corresponding to the hole (i, j)). Choosing the column j would prevent $i \geq \mu'_{k+1}$ cells to move down; hence up to a permutation of the columns, we may choose $j_s = k + s$. That is the diagram D depicted by the two shaded areas in Figure 2. The maximal number of distinct cells that can go down for that D is $\delta_k(\mu) - k - 1$. All other $D \subset \mu/ij$ such that $|D| = n - k$ will have no more than $\delta_k(\mu) - k - 1$ distinct cells that can go down, hence $e_r(\partial\bar{S})\Delta_{\mu/ij}(X_n; Y_n) = 0$ and $e_r(\bar{S}) \in I_{\mu/ij}^0$.

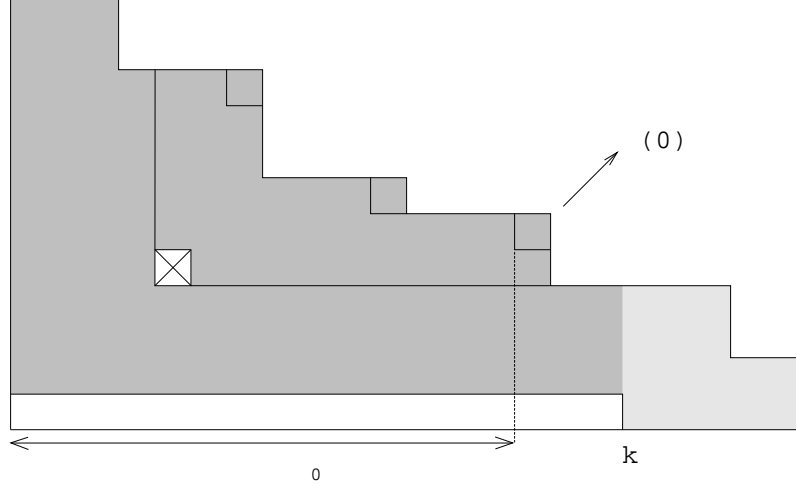


Figure 2

Finally, for $j < k \leq \beta_0$ and $e_r(\bar{S})$ as above, the only remaining problem with our argument is when the cell (i, j) is not in the darker grey area. This happens only if $i = 0$. But recall that $h_1(\partial X_n)\Delta_{\mu/ij}(X_n; Y_n) = \Delta_{\mu/i+1,j}(X_n; Y_n)$ and hence

$$e_r(\partial\bar{S})h_1(\partial X_n)\Delta_{\mu/ij}(X_n; Y_n) = e_r(\partial\bar{S})\Delta_{\mu/i+1,j}(X_n; Y_n) = 0.$$

Thus $h_1(X_n)e_r(\bar{S}) \in I_{\mu/ij}^0$ which concludes the proof of the inclusion $\tilde{I}_{\mu/ij}^0 \subseteq I_{\mu/ij}^0$.

Let $R = \mathbb{Q}[x_1, x_2, \dots, x_n]$ be the polynomial ring in n variables. The inclusion above shows that

$$\dim(R/I_{\mu/ij}^0) \leq \dim(R/\tilde{I}_{\mu/ij}^0).$$

We now present a reduction algorithm, modulo $\tilde{I}_{\mu/ij}^0$, that reduces any basis element of R as a linear combination of the basis elements of $R/I_{\mu/ij}^0$. This will show that

$$(5.2) \quad \dim(R/\tilde{I}_{\mu/ij}^0) \leq \dim(R/I_{\mu/ij}^0)$$

and conclude the proof of the Theorem 5.1.

From classical invariant theory, the set

$$(5.3) \quad \{h_\lambda(X_n)x_1^{\epsilon_1}x_2^{\epsilon_2}\cdots x_{n-1}^{\epsilon_{n-1}} : 0 \leq \epsilon_i \leq i-1\}$$

we can use Lemma 4.2 and

$$h_r(S) \equiv (-1)^r e_r(\overline{S}) \pmod{\left(\widetilde{I}_{\mu/ij}^0 + \langle h_1(X_n) \rangle\right)},$$

where $\overline{S} = X_n - S$. In this case, $e_r(\overline{S}) \in \widehat{J}_{\mu/ij}^0 \subseteq \widetilde{I}_{\mu/ij}^0$ and thus $h_r(S) \in \widetilde{I}_{\mu/ij}^0 + \langle h_1(X_n) \rangle$. For $\zeta > 0$, according to Lemma 4.2 again, we have $h_r(S) \equiv (-1)^r e_r(\overline{S}) \pmod{I_{1^n}^0}$. In particular $h_1^\zeta(X_n)h_r(S) \equiv (-1)^r h_1^\zeta(X_n)e_r(\overline{S}) \pmod{h_1^\zeta(X_n)I_{1^n}^0}$. Since $h_1^\zeta(X_n)I_{1^n}^0 \subseteq \widetilde{I}_{\mu/ij}^0 + \langle h_1^{\zeta+1}(X_n) \rangle$, we have

$$h_1^\zeta(X_n)h_r(S) \equiv (-1)^r h_1^\zeta e_r(\overline{S}) \pmod{\left(\widetilde{I}_{\mu/ij}^0 + \langle h_1^{\zeta+1}(X_n) \rangle\right)}.$$

In this case, $h_1(X_n)e_r(\overline{S}) \in \widehat{J}_{\mu/ij}^0 \subseteq \widetilde{I}_{\mu/ij}^0$ and thus $h_1^\zeta(X_n)h_r(S) \in \widetilde{I}_{\mu/ij}^0 + \langle h_1^{\zeta+1}(X_n) \rangle$. \square

Lemma 5.4. *Modulo $\widetilde{I}_{\mu/ij}^0$, any element of the form $h_\lambda(X_n)x_1^{\epsilon_1}x_2^{\epsilon_2}\cdots x_{n-1}^{\epsilon_{n-1}}$ with $0 \leq \epsilon_i \leq i-1$ is a linear combination of elements in $\mathcal{B}_{\mu/ij}$.*

Proof. We remark that $\langle h_1^{\ell+1}(X_n), h_2(X_n), h_3(X_n), \dots \rangle \subseteq \widetilde{I}_{\mu/ij}^0$. Hence

$$(5.5) \quad x_1^{\epsilon_1}x_2^{\epsilon_2}\cdots x_{n-1}^{\epsilon_{n-1}} \equiv 0 \pmod{\widetilde{I}_{\mu/ij}^0}$$

unless $h_\lambda(X_n) = h_1^\zeta(X_n)$ for $0 \leq \zeta \leq \ell$. We then proceed by induction on ζ , from $\zeta = \ell + 1$ down to $\zeta = 0$. The result is true for $\zeta > \ell$ since $h_1^{\ell+1}(X_n) \equiv 0 \pmod{\widetilde{I}_{\mu/ij}^0}$. For $\zeta \leq \ell$ consider $h_1^\zeta(X_n)x_1^{\epsilon_1}x_2^{\epsilon_2}\cdots x_{n-1}^{\epsilon_{n-1}}$. We assume by induction that modulo $\widetilde{I}_{\mu/ij}^0$, any $h_\mu x_1^{\eta_1}x_2^{\eta_2}\cdots x_{n-1}^{\eta_{n-1}}$ with the number of parts equal to 1 in μ is $\geq \zeta + 1$ and $0 \leq \eta_i \leq i-1$, is a linear combination of elements in $\mathcal{B}_{\mu/ij}$. From our choice of $\mathcal{B}_{\nu(\zeta)}$, there exists an element A in the linear span of $\mathcal{B}_{\nu(\zeta)}$ such that $x_1^{\epsilon_1}x_2^{\epsilon_2}\cdots x_{n-1}^{\epsilon_{n-1}} \equiv A \pmod{I_{\nu(\zeta)}^0}$. Hence there is an element $B \in I_{\nu(\zeta)}^0$ such that

$$(5.6) \quad h_1^\zeta(X_n)x_1^{\epsilon_1}x_2^{\epsilon_2}\cdots x_{n-1}^{\epsilon_{n-1}} = h_1^\zeta(X_n)A + h_1^\zeta(X_n)B.$$

Here $h_1^\zeta(X_n)A$ is in the linear span of $h_1^\zeta(X_n)\mathcal{B}_{\nu(\zeta)} \subseteq \mathcal{B}_{\mu/ij}$. From Lemma 5.3, $h_1^\zeta(X_n)B \in \widetilde{I}_{\mu/ij}^0 + \langle h_1^{\zeta+1}(X_n) \rangle$. Hence

$$h_1^\zeta(X_n)B \equiv h_1^{\zeta+1}C \pmod{\widetilde{I}_{\mu/ij}^0},$$

where C is an element of R . By our induction hypothesis $h_1^{\zeta+1}C$ is in the linear span of $\mathcal{B}_{\mu/ij}$. \square

We now recall an auxiliary result from [3]. For completeness we will sketch the proof but the interested reader will find the complete details in the original paper. We have that $\dim(R/I_{\mu/ij}^0) = \dim(M_{\mu/ij}^0)$. Using this we have

Proposition 5.5.

$$\dim(R/I_{\mu/ij}^0) \geq \sum_{\zeta=0}^{\ell} |\mathcal{B}_{\nu(\zeta)}|$$

Sketch of proof. For this, we construct an explicit independent set of the right cardinality. Recall that for ν a partition of n , a standard tableau T of shape ν is a tableau that is increasing both in rows and columns. We denote by \mathbb{S}_ν the set of standard tableaux of shape ν . We also associate to each entry j , of a standard tableau T , a non-negative integer in the following manner. Let (r_j, c_j) be the position of j in T , and let k be the largest entry of T , such that $c_k = c_j + 1$ and $k < j$. We set $\gamma_T(j) = r_j - r_k$. If there is no such k , set $\gamma_T(j) = r_j + 1$.

Recall that for $0 \leq \zeta \leq \ell$ we denote by $\nu(\zeta) = \mu/\alpha_\zeta\beta_\zeta$ the partition of n obtained from μ by removing $(\alpha_\zeta, \beta_\zeta)$, the rightmost corner that is north-east of the cell $(i + \zeta, j)$. For T a standard tableau, let \mathbf{B}_T denote the set

$$\mathbf{B}_T = \{ x_1^{m_1} x_2^{m_2} \cdots x_n^{m_n} \mid 0 \leq m_s \leq \gamma_T(s) \}.$$

A basis of $M_{\nu(\zeta)}^0$, is given by

$$\mathcal{B}_{\nu(\zeta)} = \{ m(\partial X_n) \Delta_T(X_n) \mid T \in \mathbb{S}_{\nu(\zeta)}, m(X_n) \in \mathbf{B}_T \},$$

where $\Delta_T(X_n)$ is defined in 2.2.

If T is a standard tableau of shape $\nu(\zeta)$, and $0 \leq u \leq \alpha_\zeta$ an integer, we denote by $T \uparrow_{u, \beta_\zeta}$ the tableau of shape $\mu/u\beta_\zeta$, such that

$$T \uparrow_{u, \beta_\zeta}(r, c) = \begin{cases} T(r, c) & \text{if } c \neq \beta_\zeta \text{ or } r < u, \\ T(r-1, c) & \text{if } c = \beta_\zeta \text{ and } r > u. \end{cases}$$

In other words, the tableau $T \uparrow_{u, \beta_\zeta}$ is obtained from T by ‘‘sliding’’ upward by 1 the cells in column β_ζ that are on or above row u . For μ/ij , we set

$$(5.7) \quad \tilde{\mathcal{B}}_{\mu/ij} = \bigcup_{\zeta=0}^{\ell} \mathcal{A}_{i+\alpha_\zeta-\zeta, \beta_\zeta},$$

where

$$(5.8) \quad \mathcal{A}_{u, \beta_\zeta} = \{ m(\partial X_n) \Delta_{T \uparrow_{u, \beta_\zeta}}(X_n) \mid T \in \mathbb{S}_{\nu(\zeta)}, m(X_n) \in \mathbf{B}_T \}.$$

We prove that 5.7 is an independent set, using a downward recursive argument. Using $h_1(\partial X_n) \Delta_{\mu/i, j}(X_n, Y_n) = \Delta_{\mu/i+1, j}(X_n, Y_n)$ in 2.1 we obtain the following. For T a standard tableau of shape $\nu(\zeta) = \mu/\alpha_\zeta\beta_\zeta$ and $0 \leq u \leq \alpha_\zeta$, we have $h_1(\partial X_n) \Delta_{T \uparrow_{u, \beta_\zeta}}(X_n) = \Delta_{T \uparrow_{u+1, \beta_\zeta}}(X_n)$ if $u < \alpha_\zeta$, and 0 if $u = \alpha_\zeta$. It follows from Definition 5.8 that

$$h_1(\partial X_n) \mathcal{A}_{u, \beta_\zeta} = \begin{cases} \mathcal{A}_{u+1, \beta_\zeta} & \text{if } u < \alpha_\zeta, \\ \{0\} & \text{if } u = \alpha_\zeta. \end{cases}$$

We deduce, from the linear independence of $\mathcal{B}_{\nu(\zeta)} = \mathcal{A}_{\alpha_\zeta, \beta_\zeta}$, that each $\mathcal{A}_{u, \beta_\zeta}$ is independent. Applying $h_1(\partial X_n)$ in definition 5.7 we readily check that $\tilde{\mathcal{B}}_{\mu/i+1, j} = h_1(\partial X_n) \tilde{\mathcal{B}}_{\mu/i, j}$. But we know that $h_1(\partial X_n) \mathcal{A}_{\alpha_\ell, \beta_\ell} = \{0\}$, and it is clear that $\mathcal{A}_{\alpha_\ell, \beta_\ell}$ is a subset of $\tilde{\mathcal{B}}_{\mu/i, j}$. By the induction hypothesis, $\tilde{\mathcal{B}}_{\mu/i+1, j}$ is independent, and a counting argument forces the independence of $\tilde{\mathcal{B}}_{\mu/ij}$. \square

To conclude the proof of Theorem 5.1 we use Lemma 5.4 to show that

$$\dim(R/\tilde{I}_{\mu/ij}^0) \leq |\mathcal{B}_{\mu/ij}| = \sum_{\zeta=0}^{\ell} |\mathcal{B}_{\nu(\zeta)}|.$$

Then the Equation 5.2 follows from the Proposition 5.5.

Corollary 5.6. $\mathcal{B}_{\mu/ij}$ is a basis of $R/I_{\mu/ij}^0$.

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