

**Invariant and covariant
polynomials for the
(generalized) symmetric group**

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Gröbner bases.

$$X_n = x_1, x_2, \dots, x_n$$

Graded lexicographic order in $\mathbb{C}[X_n]$

Example 1

$$P = 2x_1^3 x_2 - x_1 x_2 x_3^2 + x_2^4 - 3x_2 x_3^2, \quad LM(P) = x_1^3 x_2$$

Definition 2 A set $\mathcal{G} = \{g_1, \dots, g_m\} \subset I$ is a Gröbner basis if:

$$\forall P \in I, \exists g_i \in \mathcal{G} / LM(g_i) \mid LM(P).$$

Proposition 3 Let $\{g_1, \dots, g_m\}$ be a Gröbner basis of I . Then

$$\mathcal{B} = \{X_n^{\mathbf{p}} / \forall i, LM(g_i) \nmid X_n^{\mathbf{p}}\}$$

is a basis of the quotient $\mathbb{C}[X_n]/I$.

Proof.

• **Spanning set:**

We prove that every monomial $X_n^{\mathbf{q}} \notin \mathcal{B}$ may be rewritten (modulo I) with smaller monomials.

$$\begin{aligned}\exists i, X_n^{\mathbf{q}} &= X_n^{\alpha} LM(g_i) \\ X_n^{\mathbf{q}} &= X_n^{\alpha} (LM(g_i) - g_i) + X_n^{\alpha} g_i \\ &\equiv X_n^{\alpha} (LM(g_i) - g_i) \quad [I]\end{aligned}$$

• **Independent set:**

If $\exists(\alpha_{\mathbf{p}})$ such that

$$\begin{aligned}\sum \alpha_{\mathbf{p}} X_n^{\mathbf{p}} &\equiv 0 \quad [I] \\ \sum \alpha_{\mathbf{p}} X_n^{\mathbf{p}} &\in I \\ LM\left(\sum \alpha_{\mathbf{p}} X_n^{\mathbf{p}}\right) &\in LM(I)\end{aligned}$$

Syzygies

$F = \{f_1, f_2\}$ with

$$f_1 = x_1^3 + 2x_1x_2^2 + x_2^3 \quad \text{and} \quad f_2 = x_1x_2 - x_2^2$$

$$\begin{aligned} S(f_1, f_2) &= x_2 f_1 - x_1^2 f_2 \\ &= x_2(x_1^3 + 2x_1x_2^2 + x_2^3) \\ &\quad - x_1^2(x_1x_2 - x_2^2) \\ &= 2x_1x_2^3 + x_2^4 - x_1^2x_2^2 \\ &= -x_1^2x_2^2 + 2x_1x_2^3 + x_2^4 \\ &\equiv -x_1^2x_2^2 + 2x_1x_2^3 + x_2^4 \\ &\quad + x_1x_2(x_1x_2 - x_2^2) \\ &= x_1x_2^3 + x_2^4 \\ &\equiv x_1x_2^3 + x_2^4 - x_2^2(x_1x_2 - x_2^2) \\ &= 2x_2^4 \\ &= \overline{S(f_1, f_2)}^F \end{aligned}$$

Buchberger's criterion

Theorem 4 *A set $\mathcal{G} = \{g_1, \dots, g_m\}$ is a Gröbner basis if and only if*

$$\forall i \neq j, \overline{S(g_i, g_j)}^{\mathcal{G}} = 0.$$

Proposition 5 *If f and g have disjoint leading monomials then*

$$\overline{S(f, g)}^{\{f, g\}} = 0.$$

\mathcal{S}_n : symmetric group
group of permutations of $\{1, \dots, n\}$.

$$\#\mathcal{S}_n = n!$$

Classical action of \mathcal{S}_n on $\mathbb{C}[x_1, \dots, x_n] = \mathbb{C}[X_n]$:

$$\sigma . P(x_1, \dots, x_n) = P(X_n . \sigma) = P(x_{\sigma(1)}, \dots, x_{\sigma(n)}).$$

$P \in \mathbb{C}[X_n]$ is *symmetric* iff

$$\forall \sigma \in \mathcal{S}_n, \sigma . P = P.$$

Sym_n : space of symmetric polynomials.

Bases:

$$p_2(X_3) = x_1^2 + x_2^2 + x_3^2$$

$$e_2(X_3) = x_1x_2 + x_1x_3 + x_2x_3$$

$$h_2(X_3) = x_1^2 + x_1x_2 + x_1x_3 + x_2^2 + x_2x_3 + x_3^2$$

Definition 6 Harmonic polynomials :

$$\begin{aligned} \mathbf{H}_n &= \{P \in \mathbb{C}[X_n] / \forall k > 0, e_k(\partial) P = 0\} \\ &\simeq \mathbb{C}[X_n] / \langle e_k, k > 0 \rangle. \end{aligned}$$

Theorem 7 (Artin)

$$\dim \mathbf{H}_n = n! .$$

Proof. (Gröbner bases) ($n = 4$)

$$h_1(X_4) = x_1 + x_2 + x_3 + x_4$$

$$h_2(X_4) = x_1 h_1(X_4) + h_2(x_2, x_3, x_4)$$

$$\equiv h_2(x_2, x_3, x_4)$$

$$h_3(X_4) = x_1 h_2(X_4) + h_3(x_2, x_3, x_4)$$

$$\equiv h_3(x_2, x_3, x_4)$$

$$= x_2 h_2(x_2, x_3, x_4) + h_3(x_3, x_4)$$

$$\equiv h_3(x_3, x_4)$$

$$h_4(X_4) = x_1 h_3(X_4) + h_4(x_2, x_3, x_4)$$

$$\equiv h_4(x_2, x_3, x_4)$$

$$= x_2 h_3(x_2, x_3, x_4) + h_3(x_3, x_4)$$

$$\equiv h_4(x_3, x_4)$$

$$= x_3 h_3(x_3, x_4) + h_4(x_4)$$

$$\equiv h_4(x_4)$$

Leading Monomials: x_1, x_2^2, x_3^3, x_4^4 .

In general: x_1, x_2^2, \dots, x_n^n .

This is a Gröbner basis of $\langle e_k, k > 0 \rangle$.

We get a basis of the quotient :

$$\mathcal{A}_n = \{X_n^\epsilon, 0 \leq \epsilon_i \leq i - 1\}$$

(*Artin monomials*)

Corollary 8 *Hilbert series:*

$$\begin{aligned} F_q(\mathbf{H}_n) &= \sum_k q^k \dim \pi_k(\mathbf{H}_n) \\ &= 1(1 + q) \cdots (1 + q + \cdots + q^{n-1}) \\ &= [n!]_q \end{aligned}$$

Quasi-symmetrizing action of S_n :

Example 9 $n = 5$

$$\begin{aligned} & (1, 4) \bullet (x_1^2 x_2 x_5) \\ = & (1, 4) \bullet \{x_1, x_2, x_5\}^{(2,1,1)} \\ = & \{x_4, x_2, x_5\}^{(2,1,1)}_{<} \\ = & \{x_2, x_4, x_5\}^{(2,1,1)} \\ = & x_2^2 x_4 x_5. \end{aligned}$$

Invariants: $Qsym_n \supset Sym_n$

$$\begin{aligned} & x_1^2 x_2 + x_1^2 x_3 + x_2^2 x_3 \in QSym_3 \setminus Sym_3. \\ & [2,1,0] + [2,0,1] + [0,2,1] \end{aligned}$$

Definition 10 $\alpha = (\alpha_1, \dots, \alpha_l)$ a composition of $|\alpha| = k \geq 0$,

$$M_\alpha(X_n) = \sum_{0 \leq i_1 < \dots < i_l \leq n} x_{i_1}^{\alpha_1} \cdots x_{i_l}^{\alpha_l}$$

and

$$F_\alpha(X_n) = \sum_{\beta \succeq \alpha} M_\beta(X_n).$$

Example 11 $n = 3, \alpha = (1, 2)$:

$$M_\alpha = x_1 x_2^2 + x_1 x_3^2 + x_2 x_3^2$$

$$\begin{aligned} F_\alpha &= M_{(1,2)} + M_{(1,1,1)} \\ &= x_1 x_2^2 + x_1 x_3^2 + x_2 x_3^2 + x_1 x_2 x_3 \end{aligned}$$

Example 12

$$\begin{aligned} F_{(3,2)} &= M_{(3,2)} + M_{(1,2,2)} + M_{(2,1,2)} + M_{(1,1,1,2)} \\ &+ M_{(3,1,1)} + M_{(1,2,1,1)} + M_{(2,1,1,1)} + M_{(1,1,1,1,1)}. \end{aligned}$$

Super-harmonic polynomials

$$\mathcal{I}_n = \langle F_\alpha, |\alpha| > 0 \rangle.$$

Definition 13 We define

$$Q_n = \mathbb{C}[X_n] / \mathcal{I}_n,$$

and the *super-harmonic space*

$$\mathbf{SH}_n = \{P \in \mathbb{C}[X_n] / \forall |\alpha| > 0, M_\alpha(\partial)P = 0\}.$$

Remark 14

$$Sym_n \subset QSym_n \implies \mathbf{SH}_n \subset \mathbf{H}_n.$$

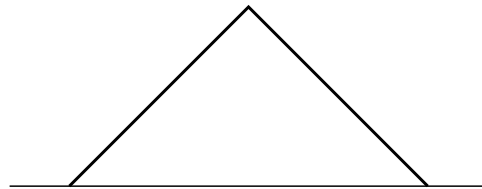
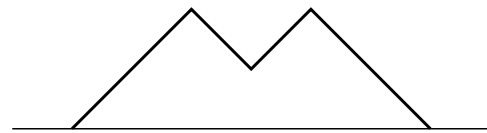
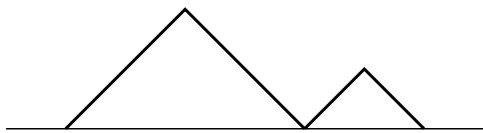
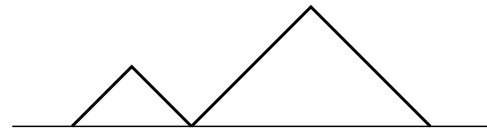
Catalan numbers

Catalan numbers

$$C_n = \frac{1}{n+1} \binom{2n}{n}$$

count (among other things !) Dyck paths.

Example 15 $n = 3$, $C_3 = 5$



Theorem 16 (Aval, Bergeron, Bergeron)

The dimension of the super-harmonic space is

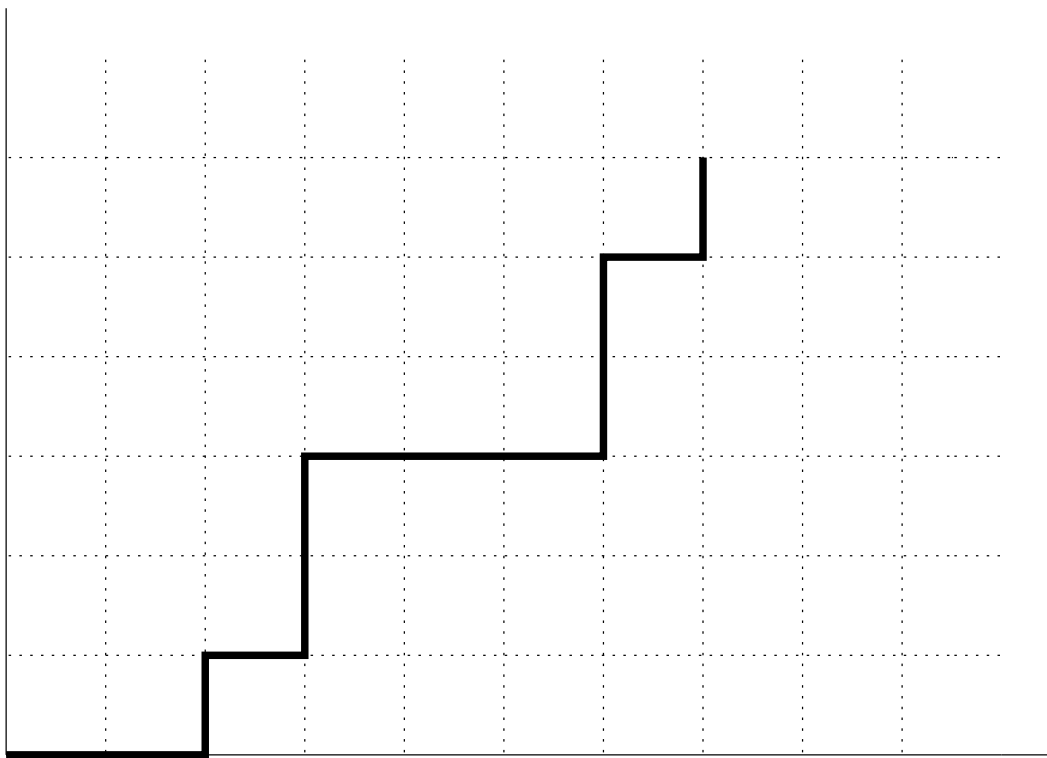
$$\dim \mathbf{SH}_n = \dim Q_n = C_n = \frac{1}{n+1} \binom{2n}{n}.$$

Coding vectors in \mathbb{N}^n by plane paths

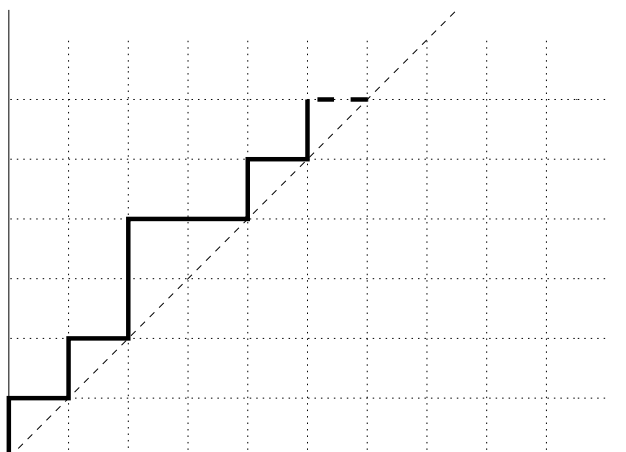
Example 17 For $n = 6$, we associate to

$$\epsilon = (2, 1, 0, 3, 0, 1)$$

the path P_ϵ :

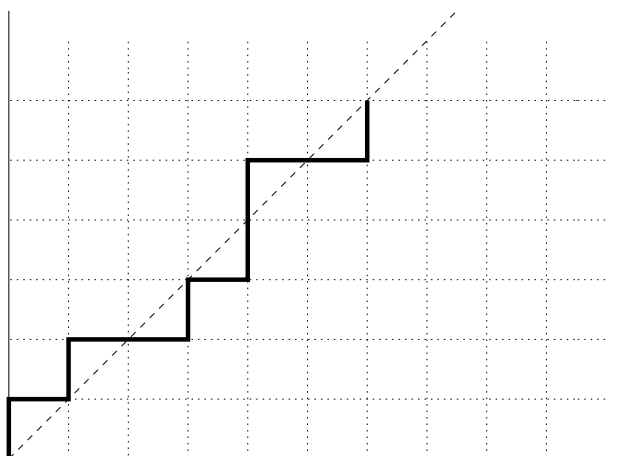


Dyck path:



$$\epsilon = (0, 1, 1, 0, 2, 1)$$

Transdiagonal path:



$$\epsilon = (0, 1, 2, 1, 0, 2)$$

Theorem 18 *The set of monomials*

$$\{X_n^\eta / \eta \text{ Dyck}\}$$

is a basis for Q_n .

Example 19 ($n = 3$)

$$F_1 = x_1 + x_2 + x_3$$

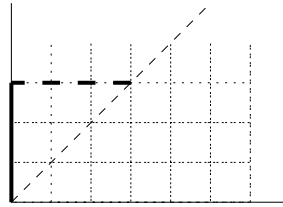
$$\begin{aligned} F_2 &= x_1^2 + x_1x_2 + x_1x_3 + x_2^2 + x_2x_3 + x_3^2 \\ &= x_1(x_1 + x_2 + x_3) + x_2^2 + x_2x_3 + x_3^2 \\ &\equiv x_2^2 + x_2x_3 + x_3^2 \end{aligned}$$

$$\begin{aligned} F_{12} &= x_1x_2^2 + x_1x_2x_3 + x_1x_3^2 + x_2x_3^2 \\ &= x_1(x_2^2 + x_2x_3 + x_3^2) + x_2x_3^2 \\ &\equiv x_2x_3^2 \end{aligned}$$

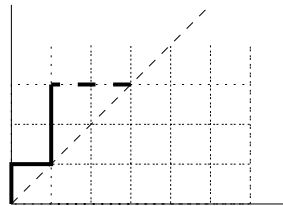
$$\begin{aligned} F_3 &= x_1F_2(x_1, x_2, x_3) + F_3(x_2, x_3) \\ &\equiv x_2^3 + x_2^2x_3 + x_2x_3^2 + x_3^3 \\ &\equiv x_2(x_2^2 + x_2x_3 + x_3^2) + x_3^3 \\ &\equiv x_3^3 \end{aligned}$$

The monomials which are not multiple of one of the $\{x_1, x_2^2, x_2x_3^2, x_3^3\}$ are:

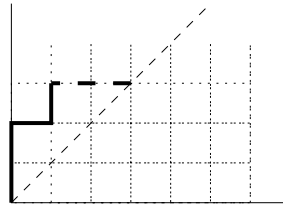
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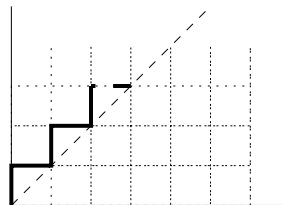
x_2



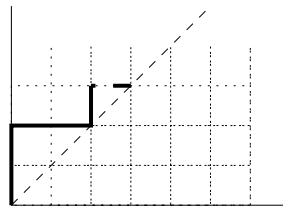
x_3



$x_2 x_3$



x_3^2



Proof.

Construction of a Gröbner basis of \mathcal{I}_n

We construct a set of polynomials

$$\mathcal{G} = \{G_\epsilon \in Q[X_n] ; \epsilon \text{ transdiagonal}\} \subset \mathcal{SI}_n$$

by induction on the length of ϵ :

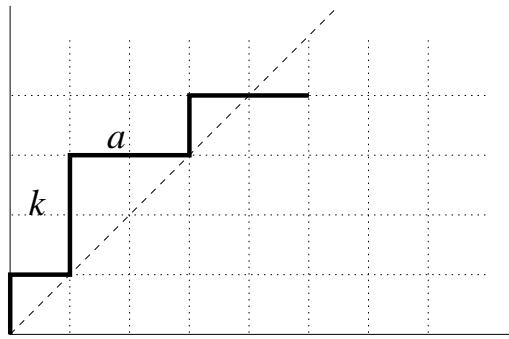
- if $\epsilon = \alpha 0^*$, then $G_\epsilon = F_\alpha$;
- otherwise, we look at the rightmost zero:

$$\epsilon = \eta 0 a \beta 0^* = \eta 0 a \beta,$$

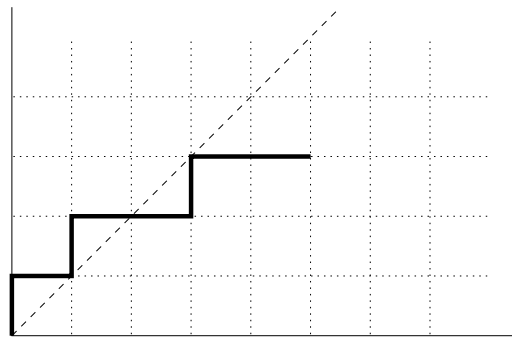
and we set (for 0 in place k):

$$G_\epsilon = G_{\eta a \beta} - x_k G_{\eta(a-1)\beta}.$$

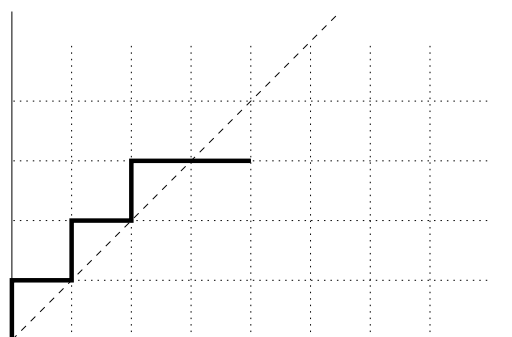
$$\varepsilon = \eta_0 a \beta$$



$$\eta a \beta$$



$$\eta(a-1)\beta$$



Example 20

$$\begin{aligned}G_{1020} &= G_{1200} - x_2 G_{1100} \\&= F_{12}(x_1, x_2, x_3, x_4) - x_2 F_{11}(x_1, x_2, x_3, x_4) \\&= x_1 x_2^2 + x_1 x_3^2 + x_1 x_4^2 + x_2 x_3^2 + x_2 x_4^2 \\&\quad + x_3 x_4^2 + x_1 x_2 x_3 + x_1 x_2 x_4 + x_1 x_3 x_4 \\&\quad + x_2 x_3 x_4 - x_2(x_1 x_2 + x_1 x_3 + x_1 x_4 \\&\quad + x_2 x_3 + x_2 x_4 + x_3 x_4) \\&= x_1 x_3^2 + x_1 x_3 x_4 + x_1 x_4^2 - x_2^2 x_3 - x_2^2 x_4 \\&\quad + x_2 x_3^2 + x_2 x_4^2 + x_3 x_4^2\end{aligned}$$

With the two properties:

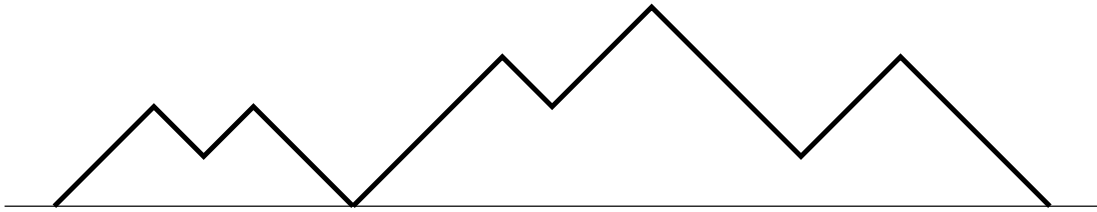
- $LM(G_\epsilon) = X_n^\epsilon$
- \mathcal{G} is a Gröbner basis of $\mathcal{I}_n = \langle F_\alpha, |\alpha| > 0 \rangle$

we prove the Theorem.

Hilbert series

$$\mathbf{SH}_n^{(k)} = \pi_k(\mathbf{SH}_n).$$

Let $C_n^{(k)}$ denote the number of Dyck paths of length $2n$ with exactly k steps down at the end.

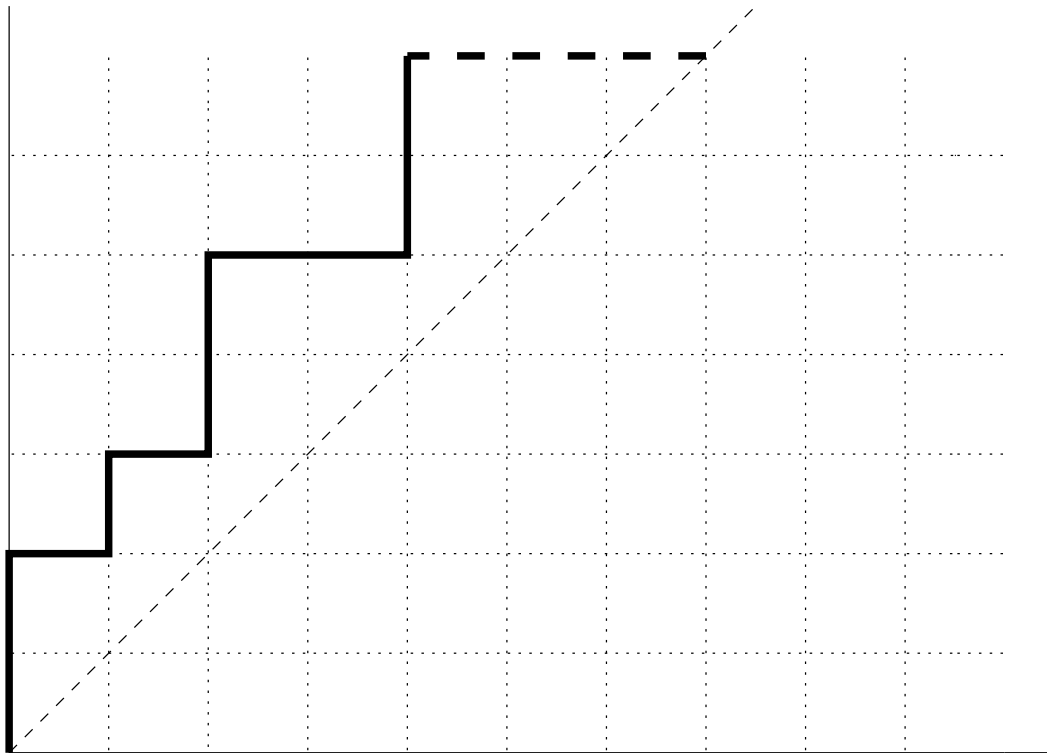


A Dyck path of length 20 with 3 final down steps.

Theorem 21

$$\dim \mathbf{SH}_n^{(k)} = C_n^{(n-k)}.$$

Example 22 For $n = 7$, the path



is associated to the monomial $x_3x_4x_6^2$ of degree
 $4 = 7 - 3$.

$$F_n(t) = \sum_{k \geq 0} \dim \mathbf{SH}_n^{(k)} t^k.$$

n	$F_n(t)$
1	1
2	$1 + t$
3	$1 + 2t + 2t^2$
4	$1 + 3t + 5t^2 + 5t^3$
5	$1 + 4t + 9t^2 + 14t^3 + 14t^4$
6	$1 + 5t + 14t^2 + 28t^3 + 42t^4 + 42t^5$
7	$1 + 6t + 20t^2 + 48t^3 + 90t^4 + 132t^5 + 132t^6$

Definition 23

$G_{n,m}$: *generalized symmetric group*

$$= C_m \wr \mathcal{S}_n$$

: group of matrices of pseudo-permutations whose non-zero entries are m -th roots of unity.

Example 24

$$\begin{pmatrix} 0 & -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix} \in G_{5,4}$$

Remark 25 $\# G_{n,m} = m^n n!$

$B_n = G_{n,2}$: *hyperoctahedral group*

: group of signed permutations.

$$\mathcal{S}_n = G_{n,1}$$

Action of $G_{n,m}$ on $\mathbb{C}[X_n]$:

$$g.P(X_n) = P(X_n.g)$$

Example 26 $m = 3$, $j = e^{\frac{i2\pi}{3}}$, $n = 3$

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & j \\ j & 0 & 0 \end{pmatrix} \cdot (x_1^2 \ x_2) = j^2 \ x_1 \ x_3^2.$$

Definition 27 Invariants :

$$P \in Inv_{n,m} \Leftrightarrow \forall g \in G_{n,m}, g.P = P.$$

Proposition 28

$$P \in Inv_{n,m} \Leftrightarrow \exists Q \in Sym_n / P(X_n) = Q(X_n^m).$$

Proof.

\Leftarrow : clear

\Rightarrow :

- $S_n \subset G_{n,m} \Rightarrow P \in Sym$

- $\mathbf{p} = (p_1, \dots, p_n)$ with $m \nmid p_j$

$$g_j = \begin{bmatrix} 1 & \dots & j-1 & j & j+1 & \dots & n \\ 1 & \dots & j-1 & \zeta j & j+1 & \dots & n \end{bmatrix}$$

$$P = \frac{1}{m} (Id + g_j + g_j^2 + \dots + g_j^{m-1}) P$$

contains no $X_n^{\mathbf{p}}$.

Definition 29 Covariant polynomials:

$$\begin{aligned} Cov_{n,m} &= \{P \in \mathbb{C}[X_n] / \forall k > 0, e_k(\partial X_n^m) P = 0\} \\ &\simeq \mathbb{C}[X_n] / \langle e_k(X_n^m), k > 0 \rangle. \end{aligned}$$

Remark 30 $Cov_{n,1} = \mathbf{H}_n$.

Theorem 31 (Chevalley)

$$\dim Cov_{n,m} = \# G_{n,m} = m^n n! .$$

Lemma 32 *Let I be an ideal of $\mathbb{C}[X_n]$*

$$I^m = \langle P(X_n^m) , P \in I \rangle.$$

$\mathcal{G} = \{g_1(X_n), \dots, g_k(X_n)\}$ *BdG of I*

\Downarrow

$\mathcal{G}^m = \{g_1(X_n^m), \dots, g_k(X_n^m)\}$ *BdG of I^m .*

Proof. Application of Buchberger's criterion :

If $\forall i \neq j$, $\overline{S(g_i, g_j)}^{\mathcal{G}} = 0$,

then $\overline{S(g_i(X_n^m), g_j(X_n^m))}^{\mathcal{G}^m} = 0$.

LM in $\langle e_k, k > 0 \rangle :$

$$x_1, x_2^2, x_3^3, \dots, x_n^n.$$

LM in $\langle e_k(X^m), k > 0 \rangle :$

$$x_1^m, x_2^{2m}, x_3^{3m}, \dots, x_n^{nm}.$$

We get a basis of the quotient:

$$\{X_n^\epsilon, 0 \leq \epsilon_i < im\}.$$

Corollary 33

$$\begin{aligned} F_q(\text{Cov}_{n,m}) &= (1 + q + \dots + q^{m-1}) F_{q^m}(\mathbf{H}_n) \\ &= [m-1]_q [n!]_{q^m} \end{aligned}$$

Quasi-symmetrizing action of $G_{n,m}$ on $\mathbb{C}[X_n]$:

$$g \bullet A^K = w(g)^{c(K)} (A \cdot |g|)_{<}^K$$

Example 34 $m = 3$, $j = e^{\frac{2i\pi}{3}}$, $n = 3$

$$\begin{aligned} & \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & j \\ j & 0 & 0 \end{pmatrix} \bullet (x_1^2 \ x_2) \\ &= (j^2)^1 \left[(x_1, x_2) \cdot \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \right]_{<}^{(2,1)} \\ &= j^2 (x_3, x_1)_{<}^{(2,1)} \\ &= j^2 (x_1, x_3)_{<}^{(2,1)} \\ &= j^2 x_1^2 x_3. \end{aligned}$$

Definition 35 Quasi-invariants:

$$P \in QInv_{n,m} \Leftrightarrow \forall g \in G_{n,m}, g \bullet P = P.$$

Proposition 36

$$P \in QInv_{n,m} \Leftrightarrow \exists Q \in QSym_n / P(X_n) = Q(X_n^m).$$

Example 37

$$\begin{aligned} M_{2,1}(X_3) &= x_1^2 x_2 + x_1^2 x_3 + x_2^2 x_3 \\ &\in QSym_3 \end{aligned}$$

$$\begin{aligned} M_{2,1}(X_3^2) &= x_1^4 x_2^2 + x_1^4 x_3^2 + x_2^4 x_3^2 \\ &\in QInv_{3,2} \end{aligned}$$

Definition 38 Super-covariants:

$$\begin{aligned} SCov_{n,m} &= \{P \in \mathbb{C}[X_n] / \forall |\alpha| > 0, M_\alpha(\partial X_n^m)P = 0\} \\ &\simeq \mathbb{C}[X_n] / \langle M_\alpha(X_n^m), |\alpha| > 0 \rangle. \end{aligned}$$

Theorem 39

$$\dim SCov_{n,m} = m^n C_n.$$

Theorem 40 *The set of monomials*

$$\{(X_n)^m \eta^{+\alpha} / \eta \text{ Dyck}, 0 \leq \alpha_i < m\}$$

is a basis of $\mathbb{C}[X_n] / \langle M_\alpha(X_n^m), |\alpha| > 0 \rangle$.

LM in $\langle M_\alpha, |\alpha| > 0 \rangle :$

$$X_n^\epsilon, \quad \epsilon \text{ transdiagonal}$$

LM in $\langle M_\alpha(X_n^m), |\alpha| > 0 \rangle :$

$$X_n^{m\epsilon}, \quad \epsilon \text{ transdiagonal}$$

We get a basis of the quotient.

Group	\mathcal{S}_n	$G_{n,m}$
Invariants	Sym_n	$Inv_{n,m}$
Covariants	$\dim \mathbf{H}_n = n!$	$\dim Cov_{n,m} = m^n n!$
Quasi-invariants	$QSym_n$	$QInv_{n,m}$
Super-covariants	$\dim \mathbf{SH}_n = C_n$	$\dim SCov_{n,m} = m^n C_n$