

**Quasi-invariant and  
super-coinvariant  
polynomials for the  
generalized symmetric group**

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## Gröbner bases.

$$X_n = x_1, x_2, \dots, x_n$$

*Graded lexicographic order* in  $\mathbb{C}[X_n]$

**Example:**

$$P = 2x_1^3 x_2 - x_1 x_2 x_3^2 + x_2^4 - 3x_2 x_3^2, \quad LM(P) = x_1^3 x_2$$

**Definition:** A set  $\mathcal{G} = \{g_1, \dots, g_m\} \subset I$  is a Gröbner basis if:

$$\forall P \in I, \quad \exists g_i \in \mathcal{G} / LM(g_i) \mid LM(P).$$

### Buchberger's criterion

A set  $\mathcal{G} = \{g_1, \dots, g_m\}$  is a Gröbner basis if and only if

$$\forall i \neq j, \quad \overline{S(g_i, g_j)}^{\mathcal{G}} = 0.$$

## Lemma

Let  $\{g_1, \dots, g_m\}$  be a Gröbner basis of  $I$ . Then

$$\mathcal{B} = \{X_n^{\mathbf{p}} \mid \forall i, LM(g_i) \nmid X_n^{\mathbf{p}}\}$$

is a basis of the quotient  $\mathbb{C}[X_n]/I$ .

# Syzygies

$F = \{f_1, f_2\}$  with

$$f_1 = x_1^3 + 2x_1x_2^2 + x_2^3 \quad \text{and} \quad f_2 = x_1x_2 - x_2^2$$

$$\begin{aligned} S(f_1, f_2) &= x_2 f_1 - x_1^2 f_2 \\ &= x_2(x_1^3 + 2x_1x_2^2 + x_2^3) \\ &\quad - x_1^2(x_1x_2 - x_2^2) \\ &= 2x_1x_2^3 + x_2^4 - x_1^2x_2^2 \\ &= -x_1^2x_2^2 + 2x_1x_2^3 + x_2^4 \\ &\equiv -x_1^2x_2^2 + 2x_1x_2^3 + x_2^4 \\ &\quad + x_1x_2(x_1x_2 - x_2^2) \\ &= x_1x_2^3 + x_2^4 \\ &\equiv x_1x_2^3 + x_2^4 - x_2^2(x_1x_2 - x_2^2) \\ &= 2x_2^4 \\ &= \overline{S(f_1, f_2)}^F \end{aligned}$$

# Quasi-symmetric polynomials

$\alpha = (\alpha_1, \dots, \alpha_l)$  a composition, we define

$$M_\alpha(X_n) = \sum_{0 \leq i_1 < \dots < i_l \leq n} x_{i_1}^{\alpha_1} \cdots x_{i_l}^{\alpha_l}$$

$$F_\alpha(X_n) = \sum_{\beta \succeq \alpha} M_\beta(X_n).$$

**Example:**  $n = 3, \alpha = (1, 2)$  :

$$M_\alpha = x_1 x_2^2 + x_1 x_3^2 + x_2 x_3^2$$

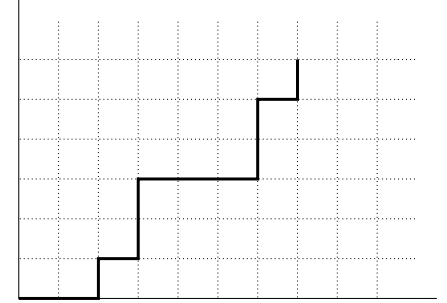
$$\begin{aligned} F_\alpha &= M_{(1,2)} + M_{(1,1,1)} \\ &= x_1 x_2^2 + x_1 x_3^2 + x_2 x_3^2 + x_1 x_2 x_3 \end{aligned}$$

$$\begin{aligned} F_{(3,2)} &= M_{(3,2)} + M_{(1,2,2)} + M_{(2,1,2)} + M_{(1,1,1,2)} \\ &+ M_{(3,1,1)} + M_{(1,2,1,1)} + M_{(2,1,1,1)} + M_{(1,1,1,1,1)}. \end{aligned}$$

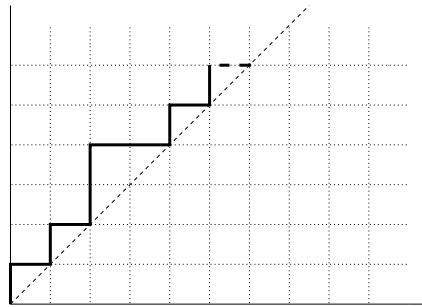
# A Gröbner basis for $QSym$

Coding vectors in  $\mathbb{N}^n$  by plane paths

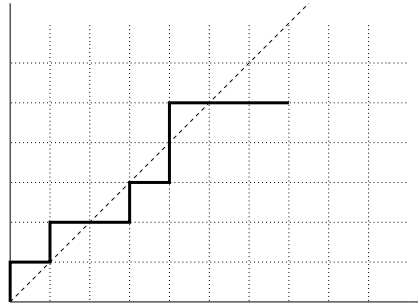
$$x_1^2 x_2 x_4^3 x_6 \longleftrightarrow \epsilon = (2, 1, 0, 3, 0, 1) \longleftrightarrow$$



Dyck path:



Transdiagonal path:



**Theorem [Aval, Bergeron, Bergeron]:**

The leading monomials of a Gröbner basis of  $\langle QSym_n \rangle$  are:

$$\{X_n^\epsilon / \epsilon \text{ transdiagonal}\}.$$

## Definitions

$G_{n,m}$  : *generalized symmetric group*

$$= C_m \wr \mathcal{S}_n$$

: group of matrices of pseudo-permutations whose non-zero entries are  $m$ -th roots of unity.

Let  $\zeta = e^{\frac{2i\pi}{m}}$ .

$$\begin{pmatrix} 0 & -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix} \in G_{5,4}$$

$$\# G_{n,m} = m^n n!$$

$B_n = G_{n,2}$  : *hyperoctahedral group*

: group of signed permutations.

$\mathcal{S}_n = G_{n,1}$  : *symmetric group*

## Classical action of $G_{n,m}$ on $\mathbb{C}[X_n]$

$$g.P(X_n) = P(X_n.g)$$

**Example:**  $m = 3, n = 3$

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & \zeta \\ \zeta & 0 & 0 \end{pmatrix} \cdot (x_1^2 \ x_2) = \zeta^2 x_1 x_3^2.$$



# Invariants

$$P \in \text{Inv}_{n,m} \Leftrightarrow \forall g \in G_{n,m}, g \cdot P = P.$$

**Remark :**  $\text{Inv}_{n,1} = \text{Sym}_n$  : symmetric functions.

## Characterization:

$$P \in \text{Inv}_{n,m} \Leftrightarrow \exists Q \in \text{Sym}_n / P(X_n) = Q(X_n^m).$$

*Proof.*

$\Leftarrow$ : clear

$\Rightarrow$ :

- $S_n \subset G_{n,m} \Rightarrow P \in \text{Sym}_n$
- $\mathbf{p} = (p_1, \dots, p_n)$  with  $m \nmid p_j$

$$g_j = \begin{bmatrix} 1 & \dots & j-1 & j & j+1 & \dots & n \\ 1 & \dots & j-1 & \zeta j & j+1 & \dots & n \end{bmatrix}$$

$$P = \frac{1}{m} (\text{Id} + g_j + g_j^2 + \dots + g_j^{m-1}) P$$

contains no  $X_n^{\mathbf{p}}$ .

## A Gröbner basis for $Sym_n$ ( $m = 1$ )

$$h_1(X_4) = x_1 + x_2 + x_3 + x_4$$

$$h_2(X_4) = x_1 h_1(X_4) + h_2(x_2, x_3, x_4)$$

$$\equiv h_2(x_2, x_3, x_4)$$

$$h_3(X_4) = x_1 h_2(X_4) + h_3(x_2, x_3, x_4)$$

$$\equiv h_3(x_2, x_3, x_4)$$

$$= x_2 h_2(x_2, x_3, x_4) + h_3(x_3, x_4)$$

$$\equiv h_3(x_3, x_4)$$

$$h_4(X_4) = \dots \equiv h_4(x_4)$$

Leading Monomials:  $x_1, x_2^2, x_3^3, x_4^4$ .

In general:  $x_1, x_2^2, \dots, x_n^n$ .

This is a Gröbner basis of  $\langle Sym_n \rangle$ .

## Useful lemma

Let  $I$  be an ideal of  $\mathbb{C}[X_n]$

$$I^m = \langle P(X_n^m), P \in I \rangle.$$

$\mathcal{G} = \{g_1(X_n), \dots, g_k(X_n)\}$  a Gröbner basis of  $I$

↓

$\mathcal{G}^m = \{g_1(X_n^m), \dots, g_k(X_n^m)\}$  a Gröbner basis of  $I^m$ .

## Proof.

Application of Buchberger's criterion :

$$\text{If } \forall i \neq j, \overline{S(g_i, g_j)}^{\mathcal{G}} = 0,$$

$$\text{then } \overline{S(g_i(X_n^m), g_j(X_n^m))}^{\mathcal{G}^m} = 0.$$

**Example:**

$$g_1 = x_1x_2 + x_1x_3 + x_2x_3, \quad g_2 = x_2^2 + x_3^2,$$
$$g_3 = x_2x_3^2 + x_3^3$$

$$\begin{aligned} S(g_1, g_2) &= x_2g_1 - x_1g_2 = x_1x_2x_3 + x_2^2x_3 - x_1x_3^2 \\ &\equiv x_1x_2x_3 + x_2^2x_3 - x_1x_3^2 - x_3g_1 \\ &\equiv x_2^2x_3 - x_3^2x_2 \\ &\equiv x_2^2x_3 - x_3^2x_2 - x_3g_2 \\ &\equiv -x_2x_3^2 - x_3^3 = -g_3 \equiv 0 \end{aligned}$$

## A Gröbner basis for $Inv_{n,m}$

A Gröbner basis when  $m = 1$  is

$$\{h_k(x_k, \dots, x_n)\}$$

and the Leading Monomials are :

$$x_1, x_2^2, x_3^3, \dots, x_n^n.$$

A Gröbner basis for  $m \in \mathbb{N}$  is

$$\{h_k(x_k^m, \dots, x_n^m)\}$$

and the Leading Monomials are

$$x_1^m, x_2^{2m}, x_3^{3m}, \dots, x_n^{nm}.$$

# Coinvariant polynomials

$$Q_{n,m} = \mathbb{C}[X_n] / \langle \text{Inv}_{n,m} \rangle$$

We get a basis of this quotient:

$$\{X_n^\epsilon \mid 0 \leq \epsilon_i < im\}.$$

**Theorem [Chevalley]:**

$$\dim Q_{n,m} = \# G_{n,m} = m^n n!.$$

**Corollary:** Hilbert series

$$F_q(Q_{n,m}) = [m]_q^n [n!]_{q^m}$$

## Quasi-symmetrizing action of $G_{n,m}$

$$g \bullet A^K = w(g)^{c(K)} (A \cdot |g|)_{<}^K$$

**Example:**  $m = 3$ ,  $\zeta = e^{\frac{2i\pi}{3}}$ ,  $n = 3$

$$\begin{aligned} & \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & \zeta \\ \zeta & 0 & 0 \end{pmatrix} \bullet (x_1^2 \ x_2) \\ &= (\zeta^2)^1 \left[ (x_1, x_2) \cdot \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \right]_{<}^{(2,1)} \\ &= \zeta^2 (x_3, x_1)_{<}^{(2,1)} \\ &= \zeta^2 (x_1, x_3)^{(2,1)} \\ &= \zeta^2 x_1^2 x_3. \end{aligned}$$

## Quasi-invariants

$$P \in QInv_{n,m} \Leftrightarrow \forall g \in G_{n,m}, g \bullet P = P.$$

**Remark:**  $QInv_{n,1} = QSym_n$  [Hivert]  
(Quasi-symmetric polynomials)

### Characterization

$$P \in QInv_{n,m} \Leftrightarrow \exists Q \in QSym_n / P(X_n) = Q(X_n^m).$$



# Super-coinvariant polynomials

$$SQ_{n,m} = \mathbb{C}[X_n] / \langle QInv_{n,m} \rangle.$$

**Theorem:**

$$\dim SQ_{n,m} = m^n C_n.$$

**Theorem:** The set of monomials

$$\{(X_n)^{m\eta+\alpha} / \eta \text{ Dyck}, 0 \leq \alpha_i < m\}$$

is a basis of  $SQ_{n,m}$ .

## A Gröbner basis for $QInv_{n,m}$

*LM* in  $\langle M_\alpha, |\alpha| > 0 \rangle :$

$$X_n^\epsilon, \quad \epsilon \text{ transdiagonal}$$

*LM* in  $\langle M_\alpha(X_n^m), |\alpha| > 0 \rangle :$

$$X_n^{m\epsilon}, \quad \epsilon \text{ transdiagonal}$$